

Ramond-Ramond Cohomology and $O(D, D)$ T-duality

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Abstract

In the name of supersymmetric double field theory, superstring effective actions can be reformulated into simple forms. They feature a pair of vielbeins corresponding to the same spacetime metric, and hence enjoy double local Lorentz symmetries. In a manifestly covariant manner—with regard to $O(D, D)$ T-duality, diffeomorphism, B -field gauge symmetry and the pair of local Lorentz symmetries—we incorporate R-R potentials into double field theory. We take them as a single object which is in a bi-fundamental spinorial representation of the double Lorentz groups. We identify cohomological structure relevant to the field strength. *A priori*, the R-R sector as well as all the fermions are $O(D, D)$ singlet. Yet, gauge fixing the two vielbeins equal to each other modifies the $O(D, D)$ transformation rule to call for a compensating local Lorentz rotation, such that the R-R potential may turn into an $O(D, D)$ spinor and T-duality can flip the chirality exchanging type IIA and IIB supergravities.

PACS: 04.60.Cf, 04.65.+e

Keywords: Ramond-Ramond sector, T-duality, Double Field Theory.

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1 Introduction

Double field theory (DFT) [1–4] may reformulate closed string effective actions *i.e.* supergravities into simple forms and manifest the $\mathbf{O}(D, D)$ T-duality which is a genuine stringy feature [5–9]. The manifestation is achieved by doubling the spacetime dimension, from D to $D + D$ with coordinates $x^\mu \rightarrow y^A = (\tilde{x}_\mu, x^\nu)$, where the newly added coordinates \tilde{x}_μ correspond to the T-dual coordinates for the closed string winding mode [10–13].

In particular, as for the $\mathbf{O}(D, D)$ covariant description of the Neveu-Schwarz (NS) sector, DFT uses $(D + D)$ -dimensional language or tensors, equipped with an $\mathbf{O}(D, D)$ invariant constant metric,

$$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (1.1)$$

Yet, DFT is not truly doubled since it is subject to a *section condition* (or “strong constraint” [3]): all the fields are required to live on a D -dimensional null hyperplane, such that the $\mathbf{O}(D, D)$ d’Alembertian

operator must be trivial acting on arbitrary fields as well as their products,

$$\partial_A \partial^A \Phi \simeq 0, \quad \partial_A \Phi_1 \partial^A \Phi_2 \simeq 0. \quad (1.2)$$

The worldsheet origin of this constraint can be traced back to the closed string level-matching condition.

Further, DFT unifies the diffeomorphism and the B -field gauge symmetry into what we may call ‘double-gauge symmetry,’ as they are generated by the generalized Lie derivative [4, 13–16],

$$\hat{\mathcal{L}}_X T_{A_1 \dots A_n} := X^B \partial_B T_{A_1 \dots A_n} + \omega_T \partial_B X^B T_{A_1 \dots A_n} + \sum_{i=1}^n (\partial_{A_i} X_B - \partial_B X_{A_i}) T_{A_1 \dots A_{i-1} B A_{i+1} \dots A_n}, \quad (1.3)$$

where ω_T is the weight of $T_{A_1 \dots A_n}$ and X^A is the double-gauge symmetry parameter whose half components are for the B -field gauge symmetry and the other half are for the diffeomorphism. Since Eq.(1.3) differs from the ordinary Lie derivative, the underlying differential geometry of DFT is not Riemannian. Namely, while doubling the spacetime dimension is sufficient to manifest the $\mathbf{O}(D, D)$ structure, the double-gauge symmetry (1.3) calls for novel mathematical treatments, such as generalized geometry [15–20], Siegel’s formalism [12, 13, 21] and our own approach [22–27] (see also [28] for a similar analysis, [29–32] for \mathcal{M} -theory extensions, and especially [33–36] for E_{11} approaches¹).

Through the series of papers [22–26], we have developed a stringy differential geometry which manifests, in a covariant manner, all the symmetries of DFT listed in Table 1. In particular, we conceived a *semi-covariant derivative* for the NS-NS sector in [22, 23], extended it to the fermionic sector [24], and managed to reformulate the $\mathcal{N} = 1$ $D = 10$ supergravity as a minimal supersymmetric double field theory (SDFT) to the full order in fermions [25]. We have also applied our formalism to construct a double field Yang-Mills theory [26].

- $\mathbf{O}(D, D)$ T-duality: *Meta-symmetry*
- Gauge symmetries
 1. Double-gauge symmetry: *Generalized Lie derivative*
 - Diffeomorphism
 - B -field gauge symmetry
 2. A pair of local Lorentz symmetries, $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$

Table 1: T-duality and gauge symmetries in DFT.

¹For recent developments related to DFT we refer to [37–60].

In this paper, within the geometric setup [22–26], as a natural next step toward the construction of $\mathcal{N} = 2$ $D = 10$ SDFT which should reformulate the type IIA and IIB supergravities in a unified manner, **we incorporate Ramond-Ramond (R-R) sector into double field theory, manifesting the $O(D, D)$ structure.** In an apparently covariant fashion, our formalism respects all the DFT symmetries listed in Table 1, including $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ the double local Lorentz symmetries. Further, our formalism does not require any specific parametrization of the DFT variables ($V_A{}^p$, $\bar{V}_A{}^{\bar{p}}$, *etc.*) in terms of the metric $g_{\mu\nu}$ and the Kalb-Ramond B -field, and is independent of the choice of the D -dimensional null hyperplane for the section condition (1.2), like $\frac{\partial}{\partial x^\mu} \simeq 0$ or $\frac{\partial}{\partial \bar{x}_\mu} \simeq 0$.

Preceding related works include the papers by Fukuma, Oota and Tanaka [61], by Hassan [62–64], by Berkovits and Howe [65], by Coimbra, Strickland-Constable and Waldram [19, 20], and by Hohm, Kwak and Zwiebach [48, 49], where the R-R sector was treated as an $O(D, D)$ spinor [48, 49, 61, 63] or as a D -dimensional bi-spinor [19, 20, 62–65]. It was pointed out by Hassan that the $O(D, D)$ transformations of the R-R sector in the two approaches are equivalent being compatible with the supersymmetry of type IIA/IIB supergravities [63]. However, while the bi-spinorial R-R field is ready to couple naturally to fermions for supersymmetry (*e.g.* the ‘democratic’ formalism [66–68] and a pure spinor approach [69]), the $O(D, D)$ spinorial R-R field appears rather awkward to do so.

In this work, we assert to put the R-R sector *a priori* in the bi-fundamental spinorial representation of $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$, rather than in the $O(D, D)$ spinorial representation. One crucial novel point in our work is that, compared to the precedents and contrary to the well-known proposition, **the R-R potential and the NS-R/R-NS fermions are *a priori* all $O(D, D)$ singlet in our covariant DFT formalism**, such that the $O(D, D)$ T-duality does not exchange type IIA and IIB supergravities! After gauge fixing the double local Lorentz symmetries to be their diagonal subgroup, the $O(D, D)$ transformation rule gets modified in order to preserve the gauge choice. Namely, the $O(D, D)$ T-duality now rotates not only the $O(D, D)$ vector indices but also local Lorentz indices: more precisely, one of the double local Lorentz indices which we choose below to be $\mathbf{Spin}(D-1, 1)_R$ rather than $\mathbf{Spin}(1, D-1)_L$ without loss of generality. That is to say, the $\mathbf{Spin}(D-1, 1)_R$ indices are no longer $O(D, D)$ singlet after the gauge fixing. In particular, the R-R potential and the NS-R fermion can flip their chiralities, resulting in the exchange of type IIA and IIB supergravities. This essentially recovers the results by Hassan [62–64] (see also *e.g.* [16, 70–75] for related recent progress). We also show that the diagonal gauge fixing may turn the R-R potential into an $O(D, D)$ spinor verifying the result by Hohm, Kwak and Zwiebach [48, 49].

A similar mechanism holds for Dirac fermions in ordinary quantum field theories on flat Minkowskian spacetime. We may gauge the internal Lorentz symmetry by introducing a spin connection which is made of a flat vielbein, and hence corresponds to a pure gauge. Then Dirac fermions are singlet for the global spacetime Lorentz symmetry. However, gauge fixing the vielbein to be trivial breaks the local Lorentz symmetry, and the fermions start to transform as a spacetime spinor under the global Lorentz symmetry. Another analogous example is the metamorphosis of the spacetime fermion into a worldsheet spinor after a gauge fixing of the kappa-symmetry in the Green-Schwarz superstring action.

The main contents as well as the organization of the present paper are as follows.

We separate out the main body into two parts. Part I is genuinely ‘double-field-theoretical’ being independent of the parametrization of the DFT variables in terms of the metric, $g_{\mu\nu}$, and the Kalb-Ramond field, $B_{\mu\nu}$. Part II deals with a specific parametrization and conveys the modified $\mathbf{O}(D, D)$ transformation rule after the diagonal gauge fixing.

1. Part I: parametrization independent formalism where R-R sector is $\mathbf{O}(D, D)$ singlet.

- Section 2 contains our covariant DFT formalism especially for R-R sector. As for a unifying description of all the R-R potentials, we consider a single bosonic object which is in a bi-fundamental spinorial representation of the double local Lorentz groups. In particular, we construct *a pair of nilpotent differential operators which can act on the R-R potential and define the field strength within the DFT formalism.*
- In section 3, we spell out the bosonic part of the type II (or $\mathcal{N} = 2$) supersymmetric double field theory Lagrangian which corresponds to the DFT reformulation of the type II democratic supergravity [66]. We derive the equations of motion and discuss the self-duality of the R-R field strength.

2. Part II: specific parametrization and gauge fixing where R-R sector is $\mathbf{O}(D, D)$ non-singlet.

- In section 4, we parametrize the covariant DFT variables in terms of a pair of D -dimensional vielbeins and a Kalb-Ramond B -field. We consider a diagonal gauge fixing of the double local Lorentz symmetries, $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$, by equating the two vielbeins. We show that, after the gauge fixing, the $\mathbf{O}(D, D)$ transformation rule must be modified to call for a compensating local Lorentz rotation. We verify that the gauge fixing may turn the R-R potential into an $\mathbf{O}(D, D)$ spinor, and further that T-duality can flip the chiralities of the R-R sector and the NS-R fermions. This manifestly realizes the exchange of type IIA and IIB supergravities.

3. Section 5 contains our conclusion.

4. In Appendix, we review in a self-contained manner the stringy differential geometry of SDFT developed in [22–26]. We set our conventions, spell out all the $\mathbf{O}(D, D)$ covariant fundamental field variables constituting type II SDFT, summarize various fully covariant quantities with respect to all the symmetries in Table 1, and discuss the reduction to ordinary Riemannian geometry.

2 R-R sector and Cohomology, before gauge fixing

2.1 R-R sector in SDFT

The NS-NS sector of SDFT consists of DFT-dilaton, d , and double-vielbeins, $V_{Ap}, \bar{V}_{A\bar{p}}$ [23–25].² For the R-R sector, we consider R-R potential, $\mathcal{C}^{\alpha}_{\bar{\alpha}}$, which is – as the indices indicate – in the *bi-fundamental spinorial representation of the double local Lorentz group*, $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ (cf. [19, 63, 64]), while being double-gauge and $\mathbf{O}(D, D)$ singlet. More precisely, ‘fundamental’ with respect to $\mathbf{Spin}(1, D-1)_L$, and ‘anti-fundamental’ with respect to $\mathbf{Spin}(D-1, 1)_R$, especially when the spinorial indices are suppressed. However, the indices can be freely lowered or raised by the symmetric charge conjugation matrices, $C_{+\alpha\beta}, \bar{C}_{+\bar{\alpha}\bar{\beta}}$ (A.2), and the distinction of being fundamental and anti-fundamental is unimportant.

The R-R potential must satisfy a ‘chirality’ condition,

$$\gamma^{(D+1)} \mathcal{C} \bar{\gamma}^{(D+1)} = \pm \mathcal{C}. \quad (2.1)$$

Hereafter, the upper sign is for type IIA and the lower sign is for type IIB.

As for the differential operators of the R-R sector, we present a pair of covariant derivatives which can be applied to any $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ bi-fundamental field, $\mathcal{T}^{\alpha}_{\bar{\beta}}$,

$$\mathcal{D}_+^0 \mathcal{T} := \gamma^A D_A^0 \mathcal{T} + \gamma^{(D+1)} D_A^0 \mathcal{T} \bar{\gamma}^A, \quad \mathcal{D}_-^0 \mathcal{T} := \gamma^A D_A^0 \mathcal{T} - \gamma^{(D+1)} D_A^0 \mathcal{T} \bar{\gamma}^A. \quad (2.2)$$

Here the superscript ‘0’ indicates that the semi-covariant derivatives assume the torsionless connection (A.31). We stress that, these differential operators are covariant with respect to all the symmetries of DFT listed in Table 1. Further, as we show below in section 2.2, they are *nilpotent*, up to the section condition (1.2),

$$(\mathcal{D}_+^0)^2 \mathcal{T} \simeq 0, \quad (\mathcal{D}_-^0)^2 \mathcal{T} \simeq 0, \quad (2.3)$$

and hence, they define cohomology.

It is worth while to note

$$\mathcal{D}_{\pm}^0 (\gamma^{(D+1)} \mathcal{T}) = -\gamma^{(D+1)} \mathcal{D}_{\mp}^0 \mathcal{T}, \quad \mathcal{D}_{\pm}^0 (\mathcal{T} \bar{\gamma}^{(D+1)}) = (\mathcal{D}_{\mp}^0 \mathcal{T}) \bar{\gamma}^{(D+1)}. \quad (2.4)$$

We define the R-R field strength using one of the nilpotent differential operators (2.2),

$$\mathcal{F} := \mathcal{D}_+^0 \mathcal{C} = \gamma^A \mathcal{D}_A^0 \mathcal{C} + \gamma^{(D+1)} \mathcal{D}_A^0 \mathcal{C} \bar{\gamma}^A. \quad (2.5)$$

²For the full field contents of type II SDFT, see Table 3 in Appendix.

This quantity is also $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ bi-fundamental, and from (2.4) it carries the opposite chirality,

$$\mathcal{F} = \mp \gamma^{(D+1)} \mathcal{F} \bar{\gamma}^{(D+1)}. \quad (2.6)$$

Further, thanks to the nilpotency (2.3), the R-R gauge symmetry is simply realized by the same differential operator,

$$\delta \mathcal{C} = \mathcal{D}_+^0 \Delta \implies \delta \mathcal{F} = \mathcal{D}_+^0 (\delta \mathcal{C}) = (\mathcal{D}_+^0)^2 \Delta \simeq 0, \quad (2.7)$$

where $\Delta^{\alpha}_{\bar{\alpha}}$ is an arbitrary gauge parameter which is in the $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ bi-fundamental representation and satisfies the same chirality condition as the field strength,

$$\Delta = \mp \gamma^{(D+1)} \Delta \bar{\gamma}^{(D+1)}. \quad (2.8)$$

The R-R sector Lagrangian, \mathcal{L}_{RR} , in type II SDFT assumes a compact form:

$$\mathcal{L}_{\text{RR}} = -\frac{1}{2} e^{-2d} \mathcal{F}^{\alpha\bar{\alpha}} \mathcal{F}_{\alpha\bar{\alpha}}. \quad (2.9)$$

Under arbitrary variations of all the elementary bosonic DFT fields, V_{Ap} , $\bar{V}_{A\bar{p}}$, d and $\mathcal{C}^{\alpha}_{\bar{\alpha}}$, the R-R sector Lagrangian transforms, up to total derivatives (\cong), from (A.51), (A.53), (A.54), (A.55), as

$$\begin{aligned} \delta \mathcal{L}_{\text{RR}} &= \delta \left(-\frac{1}{2} e^{-2d} \mathcal{F}^{\alpha\bar{\alpha}} \mathcal{F}_{\alpha\bar{\alpha}} \right) \\ &\cong e^{-2d} \left(\delta \mathcal{C} - \mathcal{C} \delta d + \frac{1}{4} V^A_p \delta V_{Aq} \gamma^{pq} \mathcal{C} + \frac{1}{2} \bar{V}^A_{\bar{p}} \delta V_{Aq} \gamma^{(D+1)} \gamma^q \mathcal{C} \bar{\gamma}^{\bar{p}} - \frac{1}{4} \bar{V}^A_{\bar{p}} \delta \bar{V}_{A\bar{q}} \mathcal{C} \bar{\gamma}^{\bar{p}\bar{q}} \right)^{\alpha\bar{\alpha}} (\mathcal{D}_-^0 \mathcal{F})_{\alpha\bar{\alpha}} \\ &\quad - \frac{1}{2} e^{-2d} \bar{V}^A_{\bar{q}} \delta V_{Ap} (\gamma^p \gamma^{(D+1)} \mathcal{F} \bar{\gamma}^{\bar{q}})^{\alpha\bar{\alpha}} \mathcal{F}_{\alpha\bar{\alpha}}. \end{aligned} \quad (2.10)$$

It is remarkable that both the chiral part of δV_{Ap} and the anti-chiral part of $\delta \bar{V}_{A\bar{p}}$, as well as the variation of the DFT-dilaton, commonly lead to nothing but the equation of motion for the R-R potential,³

$$\mathcal{D}_-^0 \mathcal{F} = \mathcal{D}_-^0 \mathcal{D}_+^0 \mathcal{C} = 0. \quad (2.11)$$

Nevertheless, as we continue to discuss below in section 3, an additional self-duality relation (3.7) needs to be imposed on the R-R field strength.

³Similar phenomena occur in $\mathcal{N} = 1$ SDFT with fermions, see Eq.(33) of Ref.[25]. The observation for the variation of the DFT-dilaton was also made by David Geissbuhler [76].

2.2 Cohomology

In this subsection, we show the nilpotency (2.3) of the differential operators, \mathcal{D}_\pm^0 , which are defined to act on an arbitrary $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ bi-fundamental spinorial field.

To sketch our proof, we set with (A.37) some notations,

$$\begin{aligned}\Phi_A^0 &= \frac{1}{4}\Phi_{Apq}^0\gamma^{pq} = \frac{1}{4}(V^B{}_p\partial_A V_{Bq} + \Gamma_{Apq}^0)\gamma^{pq}, \\ \bar{\Phi}_A^0 &= \frac{1}{4}\bar{\Phi}_{A\bar{p}\bar{q}}^0\bar{\gamma}^{\bar{p}\bar{q}} = \frac{1}{4}(\bar{V}^B{}_{\bar{p}}\partial_A \bar{V}_{B\bar{q}} + \Gamma_{A\bar{p}\bar{q}}^0)\bar{\gamma}^{\bar{p}\bar{q}}, \\ F_{AB}^0 &= \partial_A\Phi_B^0 - \partial_B\Phi_A^0 + [\Phi_A^0, \Phi_B^0] = \frac{1}{4}F_{ABpq}^0\gamma^{pq}, \\ \bar{F}_{AB}^0 &= \partial_A\bar{\Phi}_B^0 - \partial_B\bar{\Phi}_A^0 + [\bar{\Phi}_A^0, \bar{\Phi}_B^0] = \frac{1}{4}\bar{F}_{AB\bar{p}\bar{q}}^0\bar{\gamma}^{\bar{p}\bar{q}}.\end{aligned}\tag{2.12}$$

Also, if no confusion arises, we may convert an $\mathbf{O}(D, D)$ vector index either to a $\mathbf{Spin}(1, D-1)_L$ or to a $\mathbf{Spin}(D-1, 1)_R$ vector index via contraction with the double-vielbein, $V^A{}_p$ or $\bar{V}^A{}_{\bar{p}}$ respectively, such as (A.63), (A.72).

Without loss of generality, for simplicity we consider an arbitrary bi-fundamental spinor, $\mathcal{T}^\alpha_{\bar{\alpha}}$, which has zero weight. We begin with the expression,

$$(\mathcal{D}_\pm^0)^2\mathcal{T} = \mathcal{D}^{0A}\mathcal{D}_A^0\mathcal{T} + \frac{1}{2}\gamma^{AB}[\mathcal{D}_A^0, \mathcal{D}_B^0]\mathcal{T} - \frac{1}{2}[\mathcal{D}_A^0, \mathcal{D}_B^0]\mathcal{T}\bar{\gamma}^{AB} \mp \gamma^{(D+1)}\gamma^A[\mathcal{D}_A^0, \mathcal{D}_B^0]\mathcal{T}\bar{\gamma}^B, \tag{2.13}$$

into which we need to substitute

$$\begin{aligned}[\mathcal{D}_A^0, \mathcal{D}_B^0]\mathcal{T} &= -\Gamma^{0C}{}_{AB}\mathcal{D}_C^0\mathcal{T} + F_{AB}^0\mathcal{T} - \mathcal{T}\bar{F}_{AB}^0 \\ &= -\Gamma^{0C}{}_{AB}\partial_C\mathcal{T} + (F_{AB}^0 - \Gamma^{0C}{}_{AB}\Phi_C^0)\mathcal{T} - \mathcal{T}(\bar{F}_{AB}^0 - \Gamma^{0C}{}_{AB}\bar{\Phi}_C^0),\end{aligned}\tag{2.14}$$

and

$$\begin{aligned}\mathcal{D}^{0A}\mathcal{D}_A^0\mathcal{T} &\simeq (\partial^A\Phi_A^0 - \Phi^{0A}\Phi_A^0 + \Gamma_A{}^{AB}\Phi_B^0)\mathcal{T} - \mathcal{T}(\partial^A\bar{\Phi}_A^0 + \bar{\Phi}^{0A}\bar{\Phi}_A^0 + \Gamma_A{}^{AB}\bar{\Phi}_B^0) \\ &\quad + 2\Phi^{0A}\mathcal{T}\bar{\Phi}_A^0 + 2\Phi^{0A}\mathcal{D}_A^0\mathcal{T} - 2\mathcal{D}_A^0\mathcal{T}\bar{\Phi}^{0A}.\end{aligned}\tag{2.15}$$

The first three terms on the right hand side of the equality in (2.13) then give

$$\begin{aligned}\mathcal{D}^{0A}\mathcal{D}_A^0\mathcal{T} + \frac{1}{2}\gamma^{AB}[\mathcal{D}_A^0, \mathcal{D}_B^0]\mathcal{T} - \frac{1}{2}[\mathcal{D}_A^0, \mathcal{D}_B^0]\mathcal{T}\bar{\gamma}^{AB} \\ \simeq \left[\partial_A\Phi^{0A} + \Phi_A^0\Phi^{0A} + \frac{1}{2}\gamma^{AB}F_{AB}^0 + \left(\Gamma^{0B}{}_{BA} - \frac{1}{2}\Gamma_{Apq}^0\gamma^{pq}\right)\Phi^{0A}\right]\mathcal{T} \\ - \mathcal{T}\left[\partial_A\bar{\Phi}^{0A} - \bar{\Phi}_A^0\bar{\Phi}^{0A} - \frac{1}{2}\bar{F}_{AB}^0\bar{\gamma}^{AB} + \bar{\Phi}_A^0\left(\Gamma^{0B}{}_{BA} + \frac{1}{2}\Gamma_{A\bar{p}\bar{q}}^0\bar{\gamma}^{\bar{p}\bar{q}}\right)\right] \\ - 2\Phi^{0A}\mathcal{T}\bar{\Phi}_A^0 - \frac{1}{2}\left(F_{\bar{p}\bar{q}}^0 - \Gamma^{0C}{}_{\bar{p}\bar{q}}\Phi_C^0\right)\mathcal{T}\bar{\gamma}^{\bar{p}\bar{q}} - \frac{1}{2}\gamma^{pq}\mathcal{T}\left(\bar{F}_{pq}^0 - \Gamma^{0C}{}_{pq}\bar{\Phi}_C^0\right).\end{aligned}\tag{2.16}$$

Due to the following identities which can be shown by brute force computation,

$$\begin{aligned}\partial_A \Phi^{0A} + \Phi_A^0 \Phi^{0A} + \frac{1}{2} \gamma^{AB} F_{AB}^0 + \left(\Gamma^0{}^B{}_{BA} - \frac{1}{2} \Gamma^0{}_{Apq} \gamma^{pq} \right) \Phi^{0A} &\simeq -\frac{1}{4} S_{ABCD}^0 P^{AC} P^{BD}, \\ \partial_A \bar{\Phi}^{0A} - \bar{\Phi}_A^0 \bar{\Phi}^{0A} - \frac{1}{2} \bar{F}_{AB}^0 \bar{\gamma}^{AB} + \bar{\Phi}_A^0 \left(\Gamma^0{}^B{}_{BA} + \frac{1}{2} \Gamma^0{}_{\bar{p}\bar{q}} \bar{\gamma}^{\bar{p}\bar{q}} \right) &\simeq +\frac{1}{4} S_{ABCD}^0 \bar{P}^{AC} \bar{P}^{BD},\end{aligned}\tag{2.17}$$

the first two lines of the right hand side of (2.16) get simplified, and in fact from (A.73), they vanish,

$$\begin{aligned}&\left[\partial_A \Phi^{0A} + \Phi_A^0 \Phi^{0A} + \frac{1}{2} \gamma^{AB} F_{AB}^0 + \left(\Gamma^0{}^B{}_{BA} - \frac{1}{2} \Gamma^0{}_{Apq} \gamma^{pq} \right) \Phi^{0A} \right] \mathcal{T} \\ &\quad - \mathcal{T} \left[\partial_A \bar{\Phi}^{0A} - \bar{\Phi}_A^0 \bar{\Phi}^{0A} - \frac{1}{2} \bar{F}_{AB}^0 \bar{\gamma}^{AB} + \bar{\Phi}_A^0 \left(\Gamma^0{}^B{}_{BA} + \frac{1}{2} \Gamma^0{}_{\bar{p}\bar{q}} \bar{\gamma}^{\bar{p}\bar{q}} \right) \right] \\ &\simeq -\frac{1}{4} (P^{AC} P^{BD} + \bar{P}^{AC} \bar{P}^{BD}) S_{ABCD}^0 \mathcal{T} \\ &\simeq 0.\end{aligned}\tag{2.18}$$

Further, from (A.49), we have

$$F_{\bar{p}\bar{q}pq}^0 + \bar{F}_{pq\bar{p}\bar{q}}^0 = 2S_{pq\bar{p}\bar{q}}^0 + \Gamma^0{}^C{}_{pq} \Gamma^0{}_{C\bar{p}\bar{q}} \simeq \Gamma^0{}^C{}_{pq} \Gamma^0{}_{C\bar{p}\bar{q}}.\tag{2.19}$$

Consequently, with (2.19), the remaining terms of the right hand side of (2.16) vanish too,

$$\begin{aligned}&\Phi^{0A} \mathcal{T} \bar{\Phi}_A^0 + \frac{1}{4} \left(F_{\bar{p}\bar{q}}^0 - \Gamma^0{}^C{}_{\bar{p}\bar{q}} \Phi_C^0 \right) \mathcal{T} \bar{\gamma}^{\bar{p}\bar{q}} + \frac{1}{4} \gamma^{pq} \mathcal{T} \left(\bar{F}_{pq}^0 - \Gamma^0{}^C{}_{pq} \bar{\Phi}_C^0 \right) \\ &\simeq \frac{1}{16} \left(F_{\bar{p}\bar{q}pq}^0 + \bar{F}_{pq\bar{p}\bar{q}}^0 + \Phi_{pq}^{0A} \bar{\Phi}_{A\bar{p}\bar{q}}^0 - \Gamma^0{}^A{}_{pq} \bar{\Phi}_{A\bar{p}\bar{q}}^0 - \Phi_{pq}^{0A} \Gamma^0{}_{A\bar{p}\bar{q}} \right) \gamma^{pq} \mathcal{T} \bar{\gamma}^{\bar{p}\bar{q}} \\ &\simeq \frac{1}{16} \left(F_{\bar{p}\bar{q}pq}^0 + \bar{F}_{pq\bar{p}\bar{q}}^0 - \Gamma^0{}^A{}_{pq} \Gamma^0{}_{A\bar{p}\bar{q}} \right) \gamma^{pq} \mathcal{T} \bar{\gamma}^{\bar{p}\bar{q}} \\ &\simeq \frac{1}{8} S_{pq\bar{p}\bar{q}}^0 \gamma^{pq} \mathcal{T} \bar{\gamma}^{\bar{p}\bar{q}} \\ &\simeq 0.\end{aligned}\tag{2.20}$$

Thus, (2.16) vanishes completely.

Finally, in order to see the last term in (2.13) vanish, we need identities coming from (A.38), (A.40),

$$F_{p\bar{q}rs}^0 - \Gamma^0{}^C{}_{p\bar{q}} \Gamma^0{}_{Crs} = 2S_{p\bar{q}rs}^0, \quad \bar{F}_{p\bar{q}\bar{r}\bar{s}}^0 - \Gamma^0{}^C{}_{p\bar{q}} \Gamma^0{}_{C\bar{r}\bar{s}} = 2S_{p\bar{q}\bar{r}\bar{s}}^0,\tag{2.21}$$

from (A.36),

$$\Gamma^0{}^C{}_{p\bar{q}} \Phi_{Cr\bar{s}}^0 \simeq \Gamma^0{}^C{}_{p\bar{q}} \Gamma^0{}_{Crs}, \quad \Gamma^0{}^C{}_{p\bar{q}} \bar{\Phi}_{C\bar{r}\bar{s}}^0 \simeq \Gamma^0{}^C{}_{p\bar{q}} \Gamma^0{}_{C\bar{r}\bar{s}},\tag{2.22}$$

and from the Bianchi identity (A.49),

$$S_{p\bar{q}rs}^0 \gamma^p \gamma^{rs} = 2P^{AB} S_{A\bar{q}Bs}^0 \gamma^s, \quad S_{p\bar{q}\bar{r}\bar{s}}^0 \bar{\gamma}^{\bar{r}\bar{s}} \bar{\gamma}^{\bar{q}} = 2\bar{P}^{AB} S_{ApB\bar{s}}^0 \bar{\gamma}^{\bar{s}}.\tag{2.23}$$

Thanks to these identities, the last term in (2.13) gets simplified, and eventually, with the second identity in (A.49), it vanishes,

$$\begin{aligned}
\gamma^A [\mathcal{D}_A^0, \mathcal{D}_B^0] \mathcal{T} \bar{\gamma}^B &\simeq \frac{1}{4} \gamma^p [(F_{p\bar{q}rs}^0 - \Gamma^{0A}_{p\bar{q}} \Gamma_{Ars}^0) \gamma^{rs} \mathcal{T} - \mathcal{T} \gamma^{\bar{r}\bar{s}} (\bar{F}_{p\bar{q}\bar{r}\bar{s}}^0 - \Gamma^{0A}_{p\bar{q}} \Gamma_{A\bar{r}\bar{s}}^0)] \bar{\gamma}^{\bar{q}} \\
&\simeq \frac{1}{2} \gamma^p [S_{p\bar{q}rs}^0 \gamma^{rs} \mathcal{T} - \mathcal{T} \gamma^{\bar{r}\bar{s}} S_{p\bar{q}\bar{r}\bar{s}}^0] \bar{\gamma}^{\bar{q}} \\
&\simeq (P^{AB} - \bar{P}^{AB}) S_{ApB\bar{q}}^0 \gamma^p \mathcal{T} \bar{\gamma}^{\bar{q}} \\
&\simeq 0.
\end{aligned} \tag{2.24}$$

This completes our proof of the nilpotency.

In a similar fashion, the following operators,

$$\tilde{\mathcal{D}}_+^0 \mathcal{T} := \gamma^A D_A^0 \mathcal{T} \bar{\gamma}^{(D+1)} + D_A^0 \mathcal{T} \bar{\gamma}^A, \quad \tilde{\mathcal{D}}_-^0 \mathcal{T} := \gamma^A D_A^0 \mathcal{T} \bar{\gamma}^{(D+1)} - D_A^0 \mathcal{T} \bar{\gamma}^A, \tag{2.25}$$

can be also shown to be nilpotent,

$$(\tilde{\mathcal{D}}_+^0)^2 \simeq 0, \quad (\tilde{\mathcal{D}}_-^0)^2 \simeq 0. \tag{2.26}$$

However, in this work, our main interest lies in the R-R potential, $\mathcal{C}^\alpha_{\bar{\alpha}}$, which is a bi-fundamental spinor satisfying the chirality condition, $\mathcal{C} = \pm \gamma^{(D+1)} \mathcal{C} \bar{\gamma}^{(D+1)}$ (2.1). We have then

$$\tilde{\mathcal{D}}_+^0 \mathcal{C} = (\mathcal{D}_+^0 \mathcal{C}) \bar{\gamma}^{(D+1)}, \quad \tilde{\mathcal{D}}_-^0 \mathcal{C} = (\mathcal{D}_\pm^0 \mathcal{C}) \bar{\gamma}^{(D+1)}. \tag{2.27}$$

Therefore, the differential operators become degenerate. For this reason, in this paper we focus on the operators, \mathcal{D}_\pm^0 (2.2).

3 Type II Democratic Double Field Theory

Combining the NS-NS sector DFT Lagrangian (A.75) [23, 25] and the R-R sector DFT Lagrangian (2.9), we are able to spell out the bosonic part of type II or $\mathcal{N} = 2$ SDFT Lagrangian,

$$\mathcal{L}_{\text{Type II}} = \mathcal{L}_{\text{NSNS}} + \mathcal{L}_{\text{RR}} = e^{-2d} \left[\frac{1}{8} (P^{AB} P^{CD} - \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD}^0 - \frac{1}{2} \text{Tr}(\mathcal{F} \bar{\mathcal{F}}) \right], \tag{3.1}$$

where $\bar{\mathcal{F}}^\alpha_{\bar{\alpha}}$ denotes the charge conjugation,

$$\bar{\mathcal{F}} := \bar{C}_+^{-1} \mathcal{F}^T C_+, \tag{3.2}$$

and the trace is over the $\mathbf{Spin}(1, D-1)_L$ spinorial index, such that $\text{Tr}(\mathcal{F}\bar{\mathcal{F}}) = \mathcal{F}^{\alpha\bar{\alpha}}\mathcal{F}_{\alpha\bar{\alpha}}$.

Under arbitrary variations of all the bosonic fields, from (2.10), (A.44), (A.54), the Lagrangian transforms, up to total derivatives (\cong), as

$$\begin{aligned} \delta\mathcal{L}_{\text{Type II}} &\cong -\frac{1}{4}e^{-2d}\delta d \left(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD} \right) S_{ACBD}^0 \\ &\quad + \frac{1}{2}e^{-2d}\bar{V}_A{}^{\bar{q}}\delta V^{Ap} \left[S_{p\bar{q}}^0 - \text{Tr}(\gamma_p\gamma^{(D+1)}\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}}) \right] \\ &\quad + e^{-2d}\text{Tr} \left[\left(\delta\mathcal{C} - \mathcal{C}\delta d + \frac{1}{4}V^A{}_p\delta V_{Aq}\gamma^{pq}\mathcal{C} + \frac{1}{2}\bar{V}^A{}_{\bar{p}}\delta V_{Aq}\gamma^{(D+1)}\gamma^q\mathcal{C}\bar{\gamma}_{\bar{p}} - \frac{1}{4}\bar{V}^A{}_{\bar{p}}\delta V_{Aq}\mathcal{C}\bar{\gamma}^{\bar{p}\bar{q}} \right) \overline{\mathcal{D}_-\mathcal{F}} \right], \end{aligned} \quad (3.3)$$

where like (3.2), $\overline{\mathcal{D}_-\mathcal{F}} = \bar{C}_+^{-1}(\mathcal{D}_-\mathcal{F})^T C_+$. The *equations of motion* are then as follows.

- For the DFT-dilaton, d ,

$$(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD}) S_{ACBD}^0 = 0. \quad (3.4)$$

Namely the NS-NS Lagrangian vanishes on-shell, $\mathcal{L}_{\text{NSNS}} = 0$.

- For the double-vielbein, $V_A{}^p, \bar{V}_A{}^{\bar{p}}$, we have the DFT generalization of the Einstein equation,

$$S_{p\bar{q}}^0 - \text{Tr}(\gamma_p\gamma^{(D+1)}\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}}) = 0. \quad (3.5)$$

- For the R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$, the equation of motion is, as anticipated in (2.11),

$$\mathcal{D}_-\mathcal{F} = \mathcal{D}_-\mathcal{D}_+\mathcal{C} = 0. \quad (3.6)$$

However, the above type II democratic DFT Lagrangian (3.1) is supposed to be pseudo [66]: an additional self-duality relation needs to be imposed on the R-R field strength by hand,

$$\mathcal{F} = \gamma^{(D+1)}\mathcal{F} = \mp \mathcal{F}\gamma^{(D+1)} \quad : \quad \text{Self-Duality}. \quad (3.7)$$

In Eq.(3.7), the second equality holds due to the first one and the chirality (2.6). From (A.69), it is clear that the self-duality (3.7) ensures the equation of motion (3.6) to hold,

$$\mathcal{D}_-\mathcal{F} = \mathcal{D}_-\left(\gamma^{(D+1)}\mathcal{F}\right) = -\gamma^{(D+1)}\mathcal{D}_+\mathcal{F} = -\gamma^{(D+1)}(\mathcal{D}_+^0)^2\mathcal{C} \simeq 0. \quad (3.8)$$

Further, since $C_+\gamma^{(D+1)}$ is anti-symmetric, the self-duality implies that the R-R sector Lagrangian vanishes too,

$$\mathcal{L}_{\text{RR}} = -\frac{1}{2}e^{-2d}\text{Tr}(\mathcal{F}\bar{\mathcal{F}}) = -\frac{1}{2}e^{-2d}\text{Tr}(\gamma^{(D+1)}\mathcal{F}\bar{\mathcal{F}}) = 0. \quad (3.9)$$

Therefore, with (3.4), the whole Lagrangian (3.1) vanishes on-shell, $\mathcal{L}_{\text{Type II}} = 0$, up to the self-duality.

4 Parametrization and Gauge Fixing

In this section, taking specific parametrization of the double-vielbein and an $\mathbf{O}(D, D)$ element, we consider a *diagonal gauge fixing of the double local Lorentz symmetries*. We discuss the consequent modification of the $\mathbf{O}(D, D)$ transformation rule and the flipping of the chirality of the theory. We further show that the gauge fixing may map the R-R potential to an $\mathbf{O}(D, D)$ spinor. We refer readers to Appendix A.4 both for the explicit parametrization of the double-vielbein we are taking and for a self-contained review of Refs.[23, 24] on the reduction to Riemannian geometry in D dimension.

4.1 Parametrization of the $\mathbf{O}(D, D)$ rotation

We parametrize a generic $\mathbf{O}(D, D)$ group element,

$$M_A{}^B = \begin{pmatrix} \mathbf{a}^\mu{}_\nu & \mathbf{b}^{\mu\sigma} \\ \mathbf{c}_{\rho\nu} & \mathbf{d}_\rho{}^\sigma \end{pmatrix}. \quad (4.1)$$

The definition of the $\mathbf{O}(D, D)$ group,

$$M_A{}^B M_C{}^D \mathcal{J}_{BD} = \mathcal{J}_{AC}, \quad \mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (4.2)$$

implies then

$$\mathbf{a}\mathbf{b}^t + \mathbf{b}\mathbf{a}^t = 0, \quad \mathbf{c}\mathbf{d}^t + \mathbf{d}\mathbf{c}^t = 0, \quad \mathbf{a}\mathbf{d}^t + \mathbf{b}\mathbf{c}^t = 1. \quad (4.3)$$

From the vectorial $\mathbf{O}(D, D)$ transformation rule of the double-vielbein,

$$V_{Ap} \longrightarrow M_A{}^B V_{Bp}, \quad \bar{V}_{A\bar{p}} \longrightarrow M_A{}^B \bar{V}_{B\bar{p}}, \quad (4.4)$$

the parametrization of the double-vielbein (A.76) gives

$$e^{-1} \longrightarrow e^{-1} [\mathbf{a}^t + (g - B)\mathbf{b}^t], \quad \bar{e}^{-1} \longrightarrow \bar{e}^{-1} [\mathbf{a}^t - (g + B)\mathbf{b}^t], \quad (4.5)$$

such that

$$(e^{-1}\bar{e})_{p\bar{p}} \longrightarrow L_p{}^q (e^{-1}\bar{e})_{q\bar{p}}, \quad (\bar{e}^{-1}e)_{\bar{p}p} \longrightarrow \bar{L}_{\bar{p}}{}^{\bar{q}} (\bar{e}^{-1}e)_{\bar{q}p}, \quad (4.6)$$

where we set

$$L = e^{-1} [\mathbf{a}^t + (g - B)\mathbf{b}^t] [\mathbf{a}^t - (g + B)\mathbf{b}^t]^{-1} e, \quad \bar{L} = (\bar{e}^{-1}e)L^{-1}(e^{-1}\bar{e}). \quad (4.7)$$

From the considerations that $(e^{-1}\bar{e})_p{}^{\bar{p}}$ and $(\bar{e}^{-1}e)_{\bar{p}}{}^p$ themselves are local Lorentz transformations and that this property must be preserved under $\mathbf{O}(D, D)$ T-duality rotation, or alternatively from direct verification using (4.3), a crucial property of L and \bar{L} follows: they correspond to local Lorentz transformations,

$$L_p{}^r L_q{}^s \eta_{rs} = \eta_{pq}, \quad \bar{L}_{\bar{p}}{}^{\bar{r}} \bar{L}_{\bar{q}}{}^{\bar{s}} \bar{\eta}_{\bar{r}\bar{s}} = \bar{\eta}_{\bar{p}\bar{q}}. \quad (4.8)$$

Even-dimensional irreducible gamma matrices are unique up to similarity transformations, essentially due to Schur's lemma. This implies, for (4.8), (A.1), (A.79), that there must be similarity transformations, S_e satisfying

$$\bar{\gamma}^{\bar{p}}(\bar{e}^{-1}e)_{\bar{p}}{}^p = S_e^{-1}(\gamma^{(D+1)}\gamma^p)S_e, \quad \gamma^{(D+1)}\gamma^p(e^{-1}\bar{e})_p{}^{\bar{p}} = S_e\bar{\gamma}^{\bar{p}}S_e^{-1}, \quad (4.9)$$

and $S_L, S_{\bar{L}}$ satisfying

$$\gamma^q L_q{}^p = S_L^{-1}\gamma^p S_L, \quad \bar{\gamma}^{\bar{q}} \bar{L}_{\bar{q}}{}^{\bar{p}} = S_{\bar{L}}^{-1}\bar{\gamma}^{\bar{p}} S_{\bar{L}}. \quad (4.10)$$

From (4.10), (A.4), we obtain

$$\gamma^{(D+1)}S_L = \det(L) S_L \gamma^{(D+1)}, \quad \bar{\gamma}^{(D+1)}S_{\bar{L}} = \det(\bar{L}) S_{\bar{L}} \bar{\gamma}^{(D+1)}, \quad (4.11)$$

where from (4.7),

$$\det(L) = \det(\bar{L})^{-1} = \frac{\det[\mathbf{a} + \mathbf{b}(g + B)]}{\det[\mathbf{a} - \mathbf{b}(g - B)]}, \quad (4.12)$$

of which the value must be either $+1$ or -1 , since L and \bar{L} are local Lorentz transformations. Thus, if $\det(\bar{L}) = +1$, $S_{\bar{L}}$ commutes with $\bar{\gamma}^{(D+1)}$. Otherwise *i.e.* $\det(\bar{L}) = -1$, they anti-commute [24].

In fact, using (4.11), one can show that S_L and $S_{\bar{L}}$ are related by

$$S_{\bar{L}} = \begin{cases} S_e^{-1} S_L^{-1} S_e & \text{for } \det(\bar{L}) = +1 \\ S_e^{-1} \gamma^{(D+1)} S_L^{-1} S_e & \text{for } \det(\bar{L}) = -1. \end{cases} \quad (4.13)$$

For later use, we also parametrize an element of $\mathfrak{so}(D, D)$ Lie algebra: we set a $2D \times 2D$ skew-symmetric matrix,

$$h_{AB} = -h_{BA} = \begin{pmatrix} \alpha^{\mu\sigma} & -(\beta^t)^\mu{}_\rho \\ \beta_\nu{}^\sigma & \gamma_{\nu\rho} \end{pmatrix} = \begin{pmatrix} -\alpha^{\sigma\mu} & -\beta_\rho{}^\mu \\ \beta_\nu{}^\sigma & -\gamma_{\rho\nu} \end{pmatrix}, \quad (4.14)$$

where $\alpha^{\mu\nu}$ and $\gamma_{\mu\nu}$ are arbitrary $D \times D$ skew-symmetric matrices, while $\beta_\mu{}^\nu$ is a generic $D \times D$ matrix.

From $M_A{}^B \approx \delta_A{}^B + h_A{}^B$ for (4.1), we obtain

$$\begin{aligned} L_p{}^q &\approx \delta_p{}^q + \tau_p{}^q, & \tau &= -2e^{-1}g\alpha e, \\ \bar{L}_{\bar{p}}{}^{\bar{q}} &\approx \delta_{\bar{p}}{}^{\bar{q}} + \bar{\tau}_{\bar{p}}{}^{\bar{q}}, & \bar{\tau} &= +2\bar{e}^{-1}g\alpha\bar{e}, \end{aligned} \quad (4.15)$$

and hence, from (4.10),

$$S_L \approx 1 - \frac{1}{4}\tau_{pq}\gamma^{pq}, \quad S_{\bar{L}} \approx 1 - \frac{1}{4}\bar{\tau}_{\bar{p}\bar{q}}\bar{\gamma}^{\bar{p}\bar{q}}. \quad (4.16)$$

4.2 Diagonal gauge fixing and modified $\mathbf{O}(D, D)$ transformation rule

Henceforth, we consider a gauge fixing of the local Lorentz symmetries by setting

$$e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}. \quad (4.17)$$

This gauge fixing breaks the double local Lorentz symmetry groups, $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$, to its diagonal subgroup, $\mathbf{Spin}(1, D-1)_D$, which acts on the unbarred $\mathbf{Spin}(1, D-1)_L$ and the barred $\mathbf{Spin}(D-1, 1)_R$ indices simultaneously, reducing SDFT to ordinary supergravity [25].

Moreover, from (4.5), the above diagonal gauge is incompatible with the vectorial $\mathbf{O}(D, D)$ transformation rule of the double-vielbein (4.4) [23, 24]. In order to preserve the diagonal gauge, it is necessary to modify the $\mathbf{O}(D, D)$ transformation rule: the $\mathbf{O}(D, D)$ rotation must accompany a compensating local Lorentz transformation. In Table 2, we present our modified $\mathbf{O}(D, D)$ T-duality transformation rule. It contains a compensating $\mathbf{Pin}(D-1, 1)_R$ local Lorentz rotation given by $\bar{L}_{\bar{q}}{}^{\bar{p}}$ (4.7) and $S_{\bar{L}}{}^{\bar{\alpha}}{}_{\bar{\beta}}$ (4.10), which depend on both the $\mathbf{O}(D, D)$ element, M (4.1), and the NS-NS backgrounds, $g_{\mu\nu}, B_{\mu\nu}$,

$$\bar{L} = \bar{e}^{-1} [\mathbf{a}^t - (g + B)\mathbf{b}^t] [\mathbf{a}^t + (g - B)\mathbf{b}^t]^{-1} \bar{e}, \quad \bar{\gamma}^{\bar{q}} \bar{L}_{\bar{q}}{}^{\bar{p}} = S_{\bar{L}}^{-1} \bar{\gamma}^{\bar{p}} S_{\bar{L}}. \quad (4.18)$$

The modified $\mathbf{O}(D, D)$ T-duality transformation rule implies, in particular,

$$\mathcal{F} \longrightarrow \mathcal{F} S_{\bar{L}}^{-1}. \quad (4.19)$$

d	\longrightarrow	d
$V_A{}^p$	\longrightarrow	$M_A{}^B V_B{}^p$
$\bar{V}_A{}^{\bar{p}}$	\longrightarrow	$M_A{}^B \bar{V}_B{}^{\bar{q}} \bar{L}_{\bar{q}}{}^{\bar{p}}$
\mathcal{C}	\longrightarrow	$\mathcal{C} S_{\bar{L}}^{-1}$
ρ	\longrightarrow	ρ
ρ'	\longrightarrow	$S_{\bar{L}} \rho'$
$\psi_{\bar{p}}$	\longrightarrow	$(\bar{L}^{-1})_{\bar{p}}{}^{\bar{q}} \psi_{\bar{q}}$
ψ'_p	\longrightarrow	$S_{\bar{L}} \psi'_p$

(4.20)

Table 2: Modified $\mathbf{O}(D, D)$ T-duality transformation rule, after the diagonal gauge fixing (4.17). It contains an induced $\mathbf{Pin}(D-1, 1)_R$ local Lorentz rotation (4.18) (cf. [62–64] and also [73–75]).

Clearly, from (4.11), if and only if $\det(\bar{L}) = -1$, the modified $\mathbf{O}(D, D)$ rotation flips the chirality of the R-R potential (2.1) as well as those of the primed fermions, ρ' , ψ'_p (A.10), since

$$\begin{aligned}
\gamma^{(D+1)} \mathcal{C} \bar{\gamma}^{(D+1)} = \pm \mathcal{C} &\longrightarrow \gamma^{(D+1)} (\mathcal{C} S_{\bar{L}}^{-1}) \bar{\gamma}^{(D+1)} = \pm \det(\bar{L}) (\mathcal{C} S_{\bar{L}}^{-1}), \\
\bar{\gamma}^{(D+1)} \psi'_p = \pm \psi'_p &\longrightarrow \bar{\gamma}^{(D+1)} (S_{\bar{L}} \psi'_p) = \pm \det(\bar{L}) (S_{\bar{L}} \psi'_p), \\
\bar{\gamma}^{(D+1)} \rho' = \mp \rho' &\longrightarrow \bar{\gamma}^{(D+1)} (S_{\bar{L}} \rho') = \mp \det(\bar{L}) (S_{\bar{L}} \rho').
\end{aligned}
\tag{4.21}$$

Thus, the mechanism above naturally realizes the exchange of type IIA and IIB supergravities under $\mathbf{O}(D, D)$ T-duality within DFT setup, as anticipated in [24].⁴ For example, on a flat background ($g = \eta$, $B = 0$), we may set both \mathbf{a} and $\mathbf{b}g$ to be diagonal with the eigenvalues, either zero or one, in an exclusive manner such that $\mathbf{a} + \mathbf{b}g = 1$. This choice corresponds to the usual discrete T-duality along toroidal directions. In this case, we get $\det(\bar{L}) = (-1)^{\sharp_{\mathbf{a}}}$ where $\sharp_{\mathbf{a}}$ counts the number of zero eigenvalues in the matrix, \mathbf{a} , and hence the number of toroidal directions on which T-duality is performed. Thus, our formula is consistent with the well-known knowledge that performing odd number of T-duality on flat backgrounds

⁴While our analysis of the chirality change is technically based on our earlier work on DFT fermions [24], novel contributions of the present paper include its generalization to R-R sector and the realization that the diagonal gauge fixing inevitably calls for the modification of the $\mathbf{O}(D, D)$ transformation rule.

exchanges type IIA and IIB superstrings.

It is also worth while to note that, since the compensating local Lorentz rotation (4.18) explicitly depends on the parametrization of the double-vielbein in terms of $g_{\mu\nu}$ and $B_{\mu\nu}$, it appears impossible to impose the modified $\mathbf{O}(D, D)$ transformation rule, Table 2, from the beginning in the parametrization-independent covariant formalism.

For later use, we write down the modified infinitesimal $\mathbf{so}(D, D)$ transformation rule,

$$\begin{aligned}
\hat{\delta}_h d &= 0, & \hat{\delta}_h \mathcal{C} &= \frac{1}{4} \bar{\tau}_{\bar{p}\bar{q}} \mathcal{C} \bar{\gamma}^{\bar{p}\bar{q}}, \\
\hat{\delta}_h V_A^p &= h_A^B V_B^p, & \hat{\delta}_h \bar{V}_A^{\bar{p}} &= h_A^B \bar{V}_B^{\bar{p}} + \bar{V}_A^{\bar{q}} \bar{\tau}_{\bar{q}}^{\bar{p}}, \\
\hat{\delta}_h \rho &= 0, & \hat{\delta}_h \rho' &= -\frac{1}{4} \bar{\tau}_{\bar{p}\bar{q}} \bar{\gamma}^{\bar{p}\bar{q}} \rho', \\
\hat{\delta}_h \psi_{\bar{p}} &= -\bar{\tau}_{\bar{p}}^{\bar{q}} \psi_{\bar{q}}, & \hat{\delta}_h \psi'_p &= -\frac{1}{4} \bar{\tau}_{\bar{p}\bar{q}} \bar{\gamma}^{\bar{p}\bar{q}} \psi'_p,
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
\hat{\delta}_h e_\mu^p &= (\beta_\mu^\nu - B_{\mu\rho} \alpha^{\rho\nu} + g_{\mu\rho} \alpha^{\rho\nu}) e_\nu^p, \\
\hat{\delta}_h \bar{e}_\mu^{\bar{p}} &= (\beta_\mu^\nu - B_{\mu\rho} \alpha^{\rho\nu} + g_{\mu\rho} \alpha^{\rho\nu}) \bar{e}_\nu^{\bar{p}}, \\
\hat{\delta}_h B_{\mu\nu} &= \gamma_{\mu\nu} - 2\beta_{[\mu}^\rho B_{\nu]\rho} - g_{\mu\rho} \alpha^{\rho\sigma} g_{\sigma\nu} - B_{\mu\rho} \alpha^{\rho\sigma} B_{\sigma\nu}, \\
\hat{\delta}_h \phi &= \frac{1}{2} (\beta_\mu^\mu - B_{\mu\rho} \alpha^{\rho\mu}),
\end{aligned} \tag{4.23}$$

where we put, $\hat{\delta}_h = \delta_h - y^A h_A^B \partial_B$ for our short hand notation: δ_h is the actual transformation and the derivative, $y^A h_A^B \partial_B$, is for the transformation of the coordinates, $\delta_h y^A = y^B h_B^A$.

4.3 Mapping the R-R potential to an $\mathbf{O}(D, D)$ spinor

In this subsection, we show that after the diagonal gauge fixing (4.17), the modified $\mathbf{O}(D, D)$ transformation rule of the R-R potential in Table 2 actually implies that the $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ bi-fundamental R-R potential can be mapped to an $\mathbf{O}(D, D)$ spinor.

With the gauge fixing, $e_\mu^p \equiv \bar{e}_\mu^{\bar{p}}$ (4.17), from (4.7), (4.15), we have

$$\bar{L} \equiv L^{-1}, \quad \tau_p^q \equiv -\bar{\tau}_{\bar{p}}^{\bar{q}}. \tag{4.24}$$

Further, we may set

$$\eta_{pq} \equiv -\bar{\eta}_{\bar{p}\bar{q}}, \quad \bar{\gamma}^{\bar{p}} \equiv \gamma^{(D+1)}\gamma^p, \quad C_{+\alpha\beta} \equiv \bar{C}_{+\bar{\alpha}\bar{\beta}}, \quad (4.25)$$

such that, from (A.4), (A.86),

$$\bar{\gamma}^{(D+1)} \equiv -\gamma^{(D+1)}, \quad \tau_{pq}\gamma^{pq} \equiv -\bar{\tau}_{\bar{p}\bar{q}}\bar{\gamma}^{\bar{p}\bar{q}}, \quad \omega_{\mu p}{}^q \equiv \bar{\omega}_{\mu\bar{p}}{}^{\bar{q}}, \quad \omega_{\mu pq}\gamma^{pq} \equiv \bar{\omega}_{\mu\bar{p}\bar{q}}\bar{\gamma}^{\bar{p}\bar{q}}. \quad (4.26)$$

As the diagonal gauge fixing tends to eliminate the distinction of the two local Lorentz groups, we may expand the R-R potential by one sort of gamma matrices, as in the democratic formulation of the R-R sector [66–68],

$$\mathcal{C} \equiv \left(\frac{1}{2}\right)^{\frac{D+2}{4}} \sum_p' \frac{1}{p!} \mathcal{C}_{a_1 a_2 \dots a_p} \gamma^{a_1 a_2 \dots a_p}. \quad (4.27)$$

where \sum_p' denotes the odd p sum for type IIA and even p sum for type IIB.

Consequently the field strength reads

$$\mathcal{F} = \mathcal{D}_+^0 \mathcal{C} \equiv \left(\frac{1}{2}\right)^{\frac{D}{4}} \sum_p' \frac{1}{(p+1)!} \mathcal{F}_{a_1 a_2 \dots a_{p+1}} \gamma^{a_1 a_2 \dots a_{p+1}}, \quad (4.28)$$

where, up to the section choice (A.83), with $D_\mu \equiv \nabla_\mu + \omega_\mu$ from (A.84),

$$\mathcal{F}_{a_1 a_2 \dots a_p} \simeq p \left(D_{[a_1} \mathcal{C}_{a_2 \dots a_p]} - \partial_{[a_1} \phi \mathcal{C}_{a_2 \dots a_p]} \right) + \frac{p!}{3!(p-3)!} H_{[a_1 a_2 a_3} \mathcal{C}_{a_4 \dots a_p]}. \quad (4.29)$$

Similarly we also obtain

$$\mathcal{D}_-^0 \mathcal{C} \simeq \sum_p' \frac{1}{(p-1)!} \left(D^b \mathcal{C}_{b a_1 \dots a_{p-1}} - \partial^b \phi \mathcal{C}_{b a_1 \dots a_{p-1}} - \frac{1}{3!} H^{bcd} \mathcal{C}_{bcd a_1 \dots a_{p-1}} \right) \gamma^{a_1 \dots a_{p-1}}. \quad (4.30)$$

That is to say, the pair of nilpotent differential operators, \mathcal{D}_+^0 and \mathcal{D}_-^0 , reduce to an exterior derivative and its dual derivative respectively,

$$\begin{aligned} \mathcal{D}_+^0 &\implies d + (H - d\phi) \wedge, \\ \mathcal{D}_-^0 &\implies * [d + (H - d\phi) \wedge] *. \end{aligned} \quad (4.31)$$

The R-R sector Lagrangian (2.9) assumes the standard form now,

$$\mathcal{L}_{\text{RR}} = -\frac{1}{2} e^{-2d} \text{Tr}(\mathcal{F} \bar{\mathcal{F}}) \equiv -\frac{1}{2} e^{-2d} \sum_p' \frac{1}{(p+1)!} \mathcal{F}_{a_1 a_2 \dots a_{p+1}} \mathcal{F}^{a_1 a_2 \dots a_{p+1}}. \quad (4.32)$$

And the infinitesimal $\mathfrak{so}(D, D)$ transformation of the R-R potential (4.22),

$$\hat{\delta}_h \mathcal{C} = \frac{1}{4} \bar{\tau}_{\bar{a}\bar{b}} \mathcal{C} \bar{\gamma}^{\bar{a}\bar{b}} \equiv -\frac{1}{4} \tau_{ab} \mathcal{C} \gamma^{ab}, \quad (4.33)$$

gives the transformation of each p -form potential,

$$\hat{\delta}_h \mathcal{C}_{a_1 \dots a_p} = -\frac{1}{4} \tau_{[a_1 a_2} \mathcal{C}_{a_3 \dots a_p]} + \frac{1}{2} p \tau_{[a_1}{}^b \mathcal{C}_{b|a_2 \dots a_p]} + \frac{1}{4} \tau^{bc} \mathcal{C}_{bca_1 \dots a_p}. \quad (4.34)$$

We continue to consider a formal sum of the p -forms,

$$\hat{\mathcal{C}} := \sum_p' \frac{1}{p!} \mathcal{C}_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \quad \mathcal{C}_{\mu_1 \dots \mu_p} = e_{\mu_1}^{a_1} \dots e_{\mu_p}^{a_p} \mathcal{C}_{a_1 \dots a_p}, \quad (4.35)$$

and perform a field redefinition of the formal sum,

$$\hat{\mathcal{A}} := e^{-\phi} e^B \wedge \hat{\mathcal{C}}. \quad (4.36)$$

From its expansion,

$$\hat{\mathcal{A}} = \sum_p' \frac{1}{p!} \mathcal{A}_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \quad (4.37)$$

we finally define an $\mathbf{O}(D, D)$ spinor,

$$|\mathcal{A}\rangle := \sum_p' \frac{1}{p!} \mathcal{A}_{\mu_1 \mu_2 \dots \mu_p} \left(\frac{1}{\sqrt{2}} \Gamma^{\mu_1} \right) \left(\frac{1}{\sqrt{2}} \Gamma^{\mu_2} \right) \dots \left(\frac{1}{\sqrt{2}} \Gamma^{\mu_p} \right) |0\rangle. \quad (4.38)$$

Here $\frac{1}{\sqrt{2}} \Gamma^\mu$'s are the normalized creation gamma matrices [48, 49, 61]. Together with the normalized annihilation gamma matrices, $\frac{1}{\sqrt{2}} \Gamma_\mu$, they form $\mathbf{O}(D, D)$ Clifford algebra,

$$\Gamma^A = (\Gamma_\mu, \Gamma^\nu), \quad \Gamma^A \Gamma^B + \Gamma^B \Gamma^A = 2\mathcal{J}^{AB}. \quad (4.39)$$

The annihilation gamma matrices annihilate the vacuum, $\Gamma_\mu |0\rangle = 0$. It is worth while to note that, unlike the D -dimensional gamma matrices satisfying $\gamma^\mu = \gamma^p (e^{-1})_p{}^\mu$, the $\mathbf{O}(D, D)$ gamma matrices, $\Gamma^A = (\Gamma_\mu, \Gamma^\nu)$, are all constant.

From (4.23), (4.34), the infinitesimal $\mathfrak{so}(D, D)$ transformation of the p -form, $\mathcal{A}_{\mu_1 \dots \mu_p}$, follows

$$\hat{\delta} \mathcal{A}_{\mu_1 \dots \mu_p} = -\frac{1}{2} \beta_\lambda{}^\lambda \mathcal{A}_{\mu_1 \dots \mu_p} + \frac{p(p-1)}{2} \gamma_{[\mu_1 \mu_2} \mathcal{A}_{\mu_3 \dots \mu_p]} + p \beta_{[\mu_1}{}^\nu \mathcal{A}_{\nu|\mu_2 \dots \mu_p]} - \frac{1}{2} \alpha^{\nu\lambda} \mathcal{A}_{\nu\lambda\mu_1 \dots \mu_p}. \quad (4.40)$$

It is straightforward to check that, this result is equivalent to the $\mathfrak{so}(D, D)$ transformation of the $\mathbf{O}(D, D)$ spinor (4.38),

$$\hat{\delta}_h |\mathcal{A}\rangle = \frac{1}{4} h_{AB} \Gamma^{AB} |\mathcal{A}\rangle, \quad (4.41)$$

establishing the desired result,

$$\hat{\delta}_h \mathcal{C} = \frac{1}{4} \bar{\tau}_{\bar{a}\bar{b}} \mathcal{C} \bar{\gamma}^{\bar{a}\bar{b}} \iff \hat{\delta}_h |\mathcal{A}\rangle = \frac{1}{4} h_{AB} \Gamma^{AB} |\mathcal{A}\rangle. \quad (4.42)$$

This completes our proof.

5 Conclusion

In this work, we have incorporated the R-R sector into double field theory in a manifestly covariant manner, with respect to $\mathbf{O}(D, D)$ T-duality, double-gauge symmetry and the pair of local Lorentz symmetries, $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$. We put the R-R sector in the bi-fundamental spinorial representation of the double Lorentz groups, and constructed a pair of nilpotent differential operators, (2.2): one for the field strength (2.5) and other for the equation of motion (2.11).

We have spelled out the bosonic part of the type II supersymmetric double field theory Lagrangian (3.1). We presented the equations of motion (3.4), (3.5), (3.6), and analyzed the self-duality of the field strength (3.7).

A priori, in the parametrization independent covariant formalism (section 2 and Appendix A), the R-R sector and all the fermions are $\mathbf{O}(D, D)$ singlet. Yet, after gauge fixing the two vielbeins equal to each other breaks the double local Lorentz groups to the diagonal subgroup,

$$\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R \rightarrow \mathbf{Spin}(1, D-1)_D. \quad (5.1)$$

Further, it modifies the $\mathbf{O}(D, D)$ transformation rule to call for a compensating $\mathbf{Pin}(D-1, 1)_R$ rotation, (4.18), which flips the chirality of the theory, if and only if $\det(\bar{L}) = -1$ (4.12), resulting in the exchange of type IIA and IIB supergravities. The modified $\mathfrak{so}(D, D)$ transformation rule of the R-R potential can be mapped to an $\mathfrak{so}(D, D)$ rotation of a corresponding $\mathbf{O}(D, D)$ spinor (4.38), (4.42).

We emphasize that, the equivalence (4.42) between the double Lorentz bi-fundamental treatment of the R-R sector, $\mathcal{C}^\alpha_{\bar{\alpha}}$, and the $\mathbf{O}(D, D)$ spinorial treatment of it, $|\mathcal{A}\rangle$, holds only after taking the diagonal gauge fixing (4.17). The parametrization independent covariant formalism (section 2 and Appendix A) appears to prefer the former approach. As mentioned ahead, while the former may couple to fermions naturally, the latter appears rather awkward to do so. The $\mathcal{N} = 2$ supersymmetrization of the Lagrangian (3.1) remains as a future work [77].

Acknowledgements

We wish to thank David Geissbuhler and Diego Marqués for interesting discussions. We are indebted to Yoonji Suh for proofreading of the manuscript. The work was supported by the National Research Foundation of Korea (NRF) grants funded by the Korea government (MEST) with the Grant No. 2005-0049409 (CQUeST) and No. 2010-0002980.

A Review: Stringy Differential Geometry for covariant SDFT

A.1 Fundamental field variables and conventions

In Table 3, we spell out all the fundamental field variables which constitute type II (or $\mathcal{N} = 2$) SDFT. In Table 4, we summarize our conventions for indices and metrics which are being used for each representation of the symmetries in Table 1.

- Bosons

$$\begin{aligned}
 & \text{– NS-NS sector} \quad \left\{ \begin{array}{ll} \text{DFT-dilaton:} & d \\ \text{Double-vielbeins:} & V_{Ap}, \quad \bar{V}_{A\bar{p}} \end{array} \right. \\
 & \text{– R-R potential:} \quad C^{\alpha}_{\bar{\alpha}}
 \end{aligned}$$

- Fermions

$$\begin{aligned}
 & \text{– DFT-dilatinos:} \quad \rho^{\alpha}, \quad \rho'^{\bar{\alpha}} \\
 & \text{– Gravitinos:} \quad \psi^{\alpha}_{\bar{p}}, \quad \psi'^{\bar{\alpha}}_p
 \end{aligned}$$

Table 3: Fundamental field variables in type II SDFT

Index	Representation	Metric
A, B, \dots	$\mathbf{O}(D, D)$ & double-gauge vector	\mathcal{J}_{AB} in Eq.(1.1)
p, q, \dots	$\mathbf{Spin}(1, D-1)_L$ vector	$\eta_{pq} = \text{diag}(- + + \dots +)$
α, β, \dots	$\mathbf{Spin}(1, D-1)_L$ spinor	$C_{+\alpha\beta}$ in Eq.(A.2)
\bar{p}, \bar{q}, \dots	$\mathbf{Spin}(D-1, 1)_R$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \text{diag}(+ - - \dots -)$
$\bar{\alpha}, \bar{\beta}, \dots$	$\mathbf{Spin}(D-1, 1)_R$ spinor	$\bar{C}_{+\bar{\alpha}\bar{\beta}}$ in Eq.(A.2)

Table 4: Indices used for each symmetry representation and the relevant metrics that raise or lower the positions of them. *A priori*, $\mathbf{O}(D, D)$ rotates only the double-gauge vector indices (capital Roman). All the metrics have been chosen to be symmetric in this paper, *cf.* [24, 25].

Throughout the paper we focus on even D -dimensional Minkowskian spacetime which admits Majorana-Weyl spinors, and hence $D \equiv 2 \pmod{8}$, or simply $D = 10$ for the critical superstring theory.

For the two Minkowskian metrics, η_{pq} and $\bar{\eta}_{\bar{p}\bar{q}}$, we introduce separately the corresponding real gamma matrices: $(\gamma^p)^\alpha{}_\beta$ and $(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}}$ satisfying

$$\begin{aligned}\gamma^p &= (\gamma^p)^* , & \gamma^p \gamma^q + \gamma^q \gamma^p &= 2\eta^{pq} , \\ \bar{\gamma}^{\bar{p}} &= (\bar{\gamma}^{\bar{p}})^* , & \bar{\gamma}^{\bar{p}} \bar{\gamma}^{\bar{q}} + \bar{\gamma}^{\bar{q}} \bar{\gamma}^{\bar{p}} &= 2\bar{\eta}^{\bar{p}\bar{q}} .\end{aligned}\tag{A.1}$$

Their charge conjugation matrices, $C_{\pm\alpha\beta}$ and $\bar{C}_{\pm\bar{\alpha}\bar{\beta}}$, meet

$$\begin{aligned}(C_+ \gamma^{p_1 p_2 \dots p_n})_{\alpha\beta} &= (-1)^{n(n-1)/2} (C_+ \gamma^{p_1 p_2 \dots p_n})_{\beta\alpha} , \\ (\bar{C}_+ \bar{\gamma}^{\bar{p}_1 \bar{p}_2 \dots \bar{p}_n})_{\bar{\alpha}\bar{\beta}} &= (-1)^{n(n-1)/2} (\bar{C}_+ \bar{\gamma}^{\bar{p}_1 \bar{p}_2 \dots \bar{p}_n})_{\bar{\beta}\bar{\alpha}} ,\end{aligned}\tag{A.2}$$

and define the charge-conjugated spinors,⁵

$$\begin{aligned}\bar{\psi}_{\bar{p}\alpha} &= \psi_p^\beta C_{+\beta\alpha} , & \bar{\rho}_\alpha &= \rho^\beta C_{+\beta\alpha} , \\ \bar{\psi}'_{p\bar{\alpha}} &= \psi_p^{\bar{\beta}} \bar{C}_{+\bar{\beta}\bar{\alpha}} , & \bar{\rho}'_{\bar{\alpha}} &= \rho'^{\bar{\beta}} \bar{C}_{+\bar{\beta}\bar{\alpha}} .\end{aligned}\tag{A.3}$$

We also set, in order to specify the chirality of the Weyl spinors,

$$\gamma^{(D+1)} := \gamma^{012\dots D-1} , \quad \bar{\gamma}^{(D+1)} := \bar{\gamma}^{012\dots D-1} ,\tag{A.4}$$

which satisfy

$$\begin{aligned}\gamma^p \gamma^{(D+1)} + \gamma^{(D+1)} \gamma^p &= 0 , & (\gamma^{(D+1)})^2 &= 1 , \\ \bar{\gamma}^{\bar{p}} \bar{\gamma}^{(D+1)} + \bar{\gamma}^{(D+1)} \bar{\gamma}^{\bar{p}} &= 0 , & (\bar{\gamma}^{(D+1)})^2 &= 1 .\end{aligned}\tag{A.5}$$

⁵In this work, we have changed our convention in the definition of the charge-conjugation from the previous works [24, 25], such that we employ not the anti-symmetric charge conjugation matrices, $C_{-\alpha\beta} = -C_{-\beta\alpha}$ and $\bar{C}_{-\bar{\alpha}\bar{\beta}} = -\bar{C}_{-\bar{\beta}\bar{\alpha}}$ but the symmetric charge conjugation matrices, $C_{+\alpha\beta} = C_{+\beta\alpha}$ and $\bar{C}_{+\bar{\alpha}\bar{\beta}} = \bar{C}_{+\bar{\beta}\bar{\alpha}}$ (A.3), in order to reduce the number of minus signs appearing in the actual computations. They are related by

$$C_- = C_+ \gamma^{(D+1)} , \quad \bar{C}_- = \bar{C}_+ \bar{\gamma}^{(D+1)} .$$

The anti-symmetric charge conjugation matrices satisfy

$$\begin{aligned}(C_- \gamma^{p_1 p_2 \dots p_n})_{\alpha\beta} &= -(-1)^{n(n+1)/2} (C_- \gamma^{p_1 p_2 \dots p_n})_{\beta\alpha} , \\ (\bar{C}_- \bar{\gamma}^{\bar{p}_1 \bar{p}_2 \dots \bar{p}_n})_{\bar{\alpha}\bar{\beta}} &= -(-1)^{n(n+1)/2} (\bar{C}_- \bar{\gamma}^{\bar{p}_1 \bar{p}_2 \dots \bar{p}_n})_{\bar{\beta}\bar{\alpha}} .\end{aligned}$$

Specifically, the type II SDFT field variables in Table 3 satisfy the following properties.

- The DFT-dilaton gives rise to a scalar density with weight one,

$$e^{-2d}. \quad (\text{A.6})$$

- The double-vielbeins satisfy the defining properties [23]:

$$\begin{aligned} V_{Ap}V^A{}_q &= \eta_{pq}, & \bar{V}_{A\bar{p}}\bar{V}^A{}_{\bar{q}} &= \bar{\eta}_{\bar{p}\bar{q}}, \\ V_{Ap}\bar{V}^A{}_{\bar{q}} &= 0, & V_{Ap}V_B{}^p + \bar{V}_{A\bar{p}}\bar{V}_B{}^{\bar{p}} &= \mathcal{J}_{AB}. \end{aligned} \quad (\text{A.7})$$

- The R-R potential, $\mathcal{C}^\alpha_{\bar{\alpha}}$, is in the bi-fundamental spinorial representation of the local Lorentz group, $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$, and satisfies a ‘chirality’ condition,

$$\gamma^{(D+1)}\mathcal{C}\bar{\gamma}^{(D+1)} = \pm \mathcal{C}. \quad (\text{A.8})$$

The upper sign is for type IIA and the lower sign is for type IIB.

- The unprimed fermions (R-NS), $\psi_{\bar{p}}^\alpha, \rho^\alpha$, are set to be Majorana-Weyl spinors of the fixed chiralities,

$$\gamma^{(D+1)}\psi_{\bar{p}} = +\psi_{\bar{p}}, \quad \gamma^{(D+1)}\rho = -\rho, \quad (\text{A.9})$$

while the primed fermions (NS-R), $\psi'_p{}^{\bar{\alpha}}, \rho'^{\bar{\alpha}}$, are Majorana-Weyl spinors of a definite yet unfixed chirality,

$$\bar{\gamma}^{(D+1)}\psi'_p = \pm\psi'_p, \quad \bar{\gamma}^{(D+1)}\rho' = \mp\rho'. \quad (\text{A.10})$$

Again, the upper sign is for type IIA and the lower sign is for type IIB. This somewhat unconventional IIA/IIB identification, compared with (A.9), is due to the opposite signatures of the D -dimensional metrics, η and $\bar{\eta}$, we have chosen in Table 4.

The double-vielbeins then generate a pair of rank-two projections [22],

$$\begin{aligned} P_{AB} &:= V_A{}^p V_{Bp}, & P_A{}^B P_B{}^C &= P_A{}^C, \\ \bar{P}_{AB} &:= \bar{V}_A{}^{\bar{p}} \bar{V}_{B\bar{p}}, & \bar{P}_A{}^{\bar{B}} \bar{P}_{\bar{B}}{}^{\bar{C}} &= \bar{P}_A{}^{\bar{C}}, \end{aligned} \quad (\text{A.11})$$

and further a pair of rank-six projections [23],

$$\begin{aligned} \mathcal{P}_{CAB}{}^{DEF} &:= P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \frac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D}, & \mathcal{P}_{CAB}{}^{DEF} \mathcal{P}_{DEF}{}^{GHI} &= \mathcal{P}_{CAB}{}^{GHI}, \\ \bar{\mathcal{P}}_{CAB}{}^{DEF} &:= \bar{P}_C{}^{\bar{D}} \bar{P}_{[A}{}^{[\bar{E}} \bar{P}_{B]}{}^{\bar{F}]} + \frac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[\bar{E}} \bar{P}^{\bar{F}]D}, & \bar{\mathcal{P}}_{CAB}{}^{DEF} \bar{\mathcal{P}}_{DEF}{}^{GHI} &= \bar{\mathcal{P}}_{CAB}{}^{GHI}. \end{aligned} \quad (\text{A.12})$$

The rank-two projections are symmetric, orthogonal and complementary to each other,

$$P_{AB} = P_{BA}, \quad \bar{P}_{AB} = \bar{P}_{BA}, \quad P_A{}^B \bar{P}_B{}^C = 0, \quad P_A{}^B + \bar{P}_A{}^B = \delta_A{}^B, \quad (\text{A.13})$$

satisfying

$$P_A{}^B V_{Bp} = V_{Ap}, \quad \bar{P}_A{}^B \bar{V}_{B\bar{p}} = \bar{V}_{A\bar{p}}, \quad \bar{P}_A{}^B V_{Bp} = 0, \quad P_A{}^B \bar{V}_{B\bar{p}} = 0. \quad (\text{A.14})$$

The rank-six projections are symmetric and traceless,

$$\begin{aligned} \mathcal{P}_{CABDEF} &= \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]}, & \bar{\mathcal{P}}_{CABDEF} &= \bar{\mathcal{P}}_{DEFCAB} = \bar{\mathcal{P}}_{C[AB]D[EF]}, \\ \mathcal{P}^A{}_{ABDEF} &= 0, \quad P^{AB} \mathcal{P}_{ABCDEF} = 0, & \bar{\mathcal{P}}^A{}_{ABDEF} &= 0, \quad \bar{P}^{AB} \bar{\mathcal{P}}_{ABCDEF} = 0. \end{aligned} \quad (\text{A.15})$$

For simplicity, we let

$$\begin{aligned} \psi_A &:= \bar{V}_A{}^{\bar{p}} \psi_{\bar{p}}, & \psi'_A &:= V_A{}^p \psi'_p, \\ \gamma^A &:= V^A{}_p \gamma^p, & \bar{\gamma}^A &:= \bar{V}^A{}_{\bar{p}} \bar{\gamma}^{\bar{p}}, \end{aligned} \quad (\text{A.16})$$

such that

$$\begin{aligned} \bar{V}^A{}_{\bar{p}} \psi_A &= \psi_{\bar{p}}, & V^A{}_p \psi'_A &= \psi'_p, \\ V^A{}_p \psi_A &= 0, & \bar{V}^A{}_{\bar{p}} \bar{\gamma}^{\bar{p}} &= 0, \\ \{\gamma^A, \gamma^B\} &= 2P^{AB}, & \{\bar{\gamma}^A, \bar{\gamma}^B\} &= 2\bar{P}^{AB}. \end{aligned} \quad (\text{A.17})$$

A.2 Semi-covariant derivatives

For each of the DFT gauge symmetry in Table 1, we assign a corresponding connection,

- Γ_A for the double-gauge symmetry,
- Φ_A for the ‘unbarred’ local Lorentz symmetry, $\mathbf{Spin}(1, D-1)_L$,
- $\bar{\Phi}_A$ for the ‘barred’ local Lorentz symmetry, $\mathbf{Spin}(D-1, 1)_R$.

Combining all of them, we employ the *master semi-covariant derivative* [24],

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A. \quad (\text{A.18})$$

It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A, \quad D_A = \partial_A + \Phi_A + \bar{\Phi}_A, \quad (\text{A.19})$$

of which the former is the semi-covariant derivative for the double-gauge symmetry [22, 23],

$$\nabla_C T_{A_1 A_2 \dots A_n} := \partial_C T_{A_1 A_2 \dots A_n} - \omega_T \Gamma_{BC}^B T_{A_1 A_2 \dots A_n} + \sum_{i=1}^n \Gamma_{CA_i}^B T_{A_1 \dots A_{i-1} B A_{i+1} \dots A_n}. \quad (\text{A.20})$$

And the latter is the covariant derivative for the pair of local Lorentz symmetries, yet being semi-covariant under the double-gauge symmetry [24].

Firstly, the master semi-covariant derivative is compatible with all the constant metrics,

$$\begin{aligned} \mathcal{D}_A \mathcal{J}_{BC} &= \nabla_A \mathcal{J}_{BC} = \Gamma_{AB}^D \mathcal{J}_{DC} + \Gamma_{AC}^D \mathcal{J}_{BD} = 0, \\ \mathcal{D}_A \eta_{pq} &= D_A \eta_{pq} = \Phi_{Ap}{}^r \eta_{rq} + \Phi_{Aq}{}^r \eta_{pr} = 0, \\ \mathcal{D}_A \bar{\eta}_{\bar{p}\bar{q}} &= D_A \bar{\eta}_{\bar{p}\bar{q}} = \bar{\Phi}_{A\bar{p}}{}^{\bar{r}} \bar{\eta}_{\bar{r}\bar{q}} + \bar{\Phi}_{A\bar{q}}{}^{\bar{r}} \bar{\eta}_{\bar{p}\bar{r}} = 0, \\ \mathcal{D}_A C_{+\alpha\beta} &= D_A C_{+\alpha\beta} = \Phi_{A\alpha}{}^\delta C_{+\delta\beta} + \Phi_{A\beta}{}^\delta C_{+\alpha\delta} = 0, \\ \mathcal{D}_A \bar{C}_{+\bar{\alpha}\bar{\beta}} &= D_A \bar{C}_{+\bar{\alpha}\bar{\beta}} = \bar{\Phi}_{A\bar{\alpha}}{}^{\bar{\delta}} \bar{C}_{+\bar{\delta}\bar{\beta}} + \bar{\Phi}_{A\bar{\beta}}{}^{\bar{\delta}} \bar{C}_{+\bar{\alpha}\bar{\delta}} = 0, \end{aligned} \quad (\text{A.21})$$

and also with all the gamma matrices,

$$\begin{aligned} \mathcal{D}_A (\gamma^p)^\alpha{}_\beta &= D_A (\gamma^p)^\alpha{}_\beta = \Phi_A{}^p{}_q (\gamma^q)^\alpha{}_\beta + \Phi_A{}^\alpha{}_\delta (\gamma^p)^\delta{}_\beta - (\gamma^p)^\alpha{}_\delta \Phi_A{}^\delta{}_\beta = 0, \\ \mathcal{D}_A (\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}} &= D_A (\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}} = \bar{\Phi}_A{}^{\bar{p}}{}_{\bar{q}} (\bar{\gamma}^{\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}} + \bar{\Phi}_A{}^{\bar{\alpha}}{}_{\bar{\delta}} (\bar{\gamma}^{\bar{p}})^{\bar{\delta}}{}_{\bar{\beta}} - (\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\delta}} \bar{\Phi}_A{}^{\bar{\delta}}{}_{\bar{\beta}} = 0. \end{aligned} \quad (\text{A.22})$$

It follows then that the connections are all anti-symmetric,

$$\Gamma_{ABC} = -\Gamma_{ACB}, \quad (\text{A.23})$$

$$\Phi_{Apq} = -\Phi_{Aqp}, \quad \bar{\Phi}_{A\bar{p}\bar{q}} = -\bar{\Phi}_{A\bar{q}\bar{p}}, \quad (\text{A.24})$$

$$\Phi_{A\alpha\beta} = -\Phi_{A\beta\alpha}, \quad \bar{\Phi}_{A\bar{\alpha}\bar{\beta}} = -\bar{\Phi}_{A\bar{\beta}\bar{\alpha}}, \quad (\text{A.25})$$

and as usual,

$$\Phi_A{}^\alpha{}_\beta = \frac{1}{4} \Phi_{Apq} (\gamma^{pq})^\alpha{}_\beta, \quad \bar{\Phi}_A{}^{\bar{\alpha}}{}_{\bar{\beta}} = \frac{1}{4} \bar{\Phi}_{A\bar{p}\bar{q}} (\bar{\gamma}^{\bar{p}\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}. \quad (\text{A.26})$$

Secondly, the master semi-covariant derivative annihilates all the NS-NS fields,

$$\begin{aligned}
\mathcal{D}_A d &= \nabla_A d := -\frac{1}{2}e^{2d}\nabla_A(e^{-2d}) = \partial_A d + \frac{1}{2}\Gamma^B{}_{BA} = 0, \\
\mathcal{D}_A V_{Bp} &= \partial_A V_{Bp} + \Gamma_{AB}{}^C V_{Cp} + \Phi_{Ap}{}^q V_{Bq} = 0, \\
\mathcal{D}_A \bar{V}_{B\bar{p}} &= \partial_A \bar{V}_{B\bar{p}} + \Gamma_{AB}{}^C \bar{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}} \bar{V}_{B\bar{q}} = 0.
\end{aligned} \tag{A.27}$$

It follows that

$$\mathcal{D}_A P_{BC} = \nabla_A P_{BC} = 0, \quad \mathcal{D}_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0, \tag{A.28}$$

and the connections are related to each other by

$$\begin{aligned}
\Gamma_{ABC} &= V_B{}^p D_A V_{Cp} + \bar{V}_B{}^{\bar{p}} D_A \bar{V}_{C\bar{p}}, \\
\Phi_{Apq} &= V^B{}_p \nabla_A V_{Bq}, \\
\bar{\Phi}_{A\bar{p}\bar{q}} &= \bar{V}^B{}_{\bar{p}} \nabla_A \bar{V}_{B\bar{q}}.
\end{aligned} \tag{A.29}$$

The connections assume the following most general forms [25],

$$\begin{aligned}
\Gamma_{CAB} &= \Gamma_{CAB}^0 + \Delta_{Cpq} V_A{}^p V_B{}^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}}, \\
\Phi_{Apq} &= \Phi_{Apq}^0 + \Delta_{Apq}, \\
\bar{\Phi}_{A\bar{p}\bar{q}} &= \bar{\Phi}_{A\bar{p}\bar{q}}^0 + \bar{\Delta}_{A\bar{p}\bar{q}}.
\end{aligned} \tag{A.30}$$

Here, from [23],⁶

$$\begin{aligned}
\Gamma_{CAB}^0 &= 2(P\partial_C P\bar{P})_{[AB]} + 2(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E) \partial_D P_{EC} \\
&\quad - \frac{4}{D-1} (\bar{P}_{C[A} \bar{P}_{B]}{}^D + P_{C[A} P_{B]}{}^D) (\partial_D d + (P\partial^E P\bar{P})_{[ED]}),
\end{aligned} \tag{A.31}$$

and, in terms of this, with the corresponding derivative, $\nabla_A^0 = \partial_A + \Gamma_A^0$,

$$\begin{aligned}
\Phi_{Apq}^0 &= V^B{}_p \nabla_A^0 V_{Bq} = V^B{}_p \partial_A V_{Bq} + \Gamma_{ABC}^0 V^B{}_p V^C{}_q, \\
\bar{\Phi}_{A\bar{p}\bar{q}}^0 &= \bar{V}^B{}_{\bar{p}} \nabla_A^0 \bar{V}_{B\bar{q}} = \bar{V}^B{}_{\bar{p}} \partial_A \bar{V}_{B\bar{q}} + \Gamma_{ABC}^0 \bar{V}^B{}_{\bar{p}} \bar{V}^C{}_{\bar{q}}.
\end{aligned} \tag{A.32}$$

⁶For a recent rederivation of the connection (A.31) and related discussion, see [28].

Further, the extra pieces, Δ_{Apq} and $\bar{\Delta}_{A\bar{p}\bar{q}}$, correspond to the ‘torsion’ of SDFT, which must be covariant and satisfy [25]

$$\begin{aligned}\Delta_{Apq} &= -\Delta_{Aqp}, & \bar{\Delta}_{A\bar{p}\bar{q}} &= -\bar{\Delta}_{A\bar{q}\bar{p}}, \\ \Delta_{Apq}V^{Ap} &= 0, & \bar{\Delta}_{A\bar{p}\bar{q}}\bar{V}^{A\bar{p}} &= 0.\end{aligned}\tag{A.33}$$

Otherwise they are arbitrary. That is to say, with (A.31), (A.32), (A.33), the connections in (A.30) provide the most general solution to the constraints, (A.21), (A.22) and (A.27). Note that, the latter two ‘traceless’ conditions in (A.33) are necessary in order to maintain $\mathcal{D}_A d = 0$. As is the case in ordinary supergravities, the torsion can be constructed from the bi-spinorial objects. We refer to our earlier work [25] for the torsions in the case of $\mathcal{N} = 1$ $D = 10$ SDFT.

The torsionless connection, Γ_{ABC}^0 (A.31), further obeys [23, 24],

$$\Gamma_{ABC}^0 + \Gamma_{BCA}^0 + \Gamma_{CAB}^0 = 0, \tag{A.34}$$

and

$$\mathcal{P}_{CAB}{}^{DEF}\Gamma_{DEF}^0 = 0, \quad \bar{\mathcal{P}}_{CAB}{}^{DEF}\Gamma_{DEF}^0 = 0. \tag{A.35}$$

In fact, the torsionless connection, Γ_{ABC}^0 (A.31), is the unique connection which satisfies these extra properties: enforcing (A.34) and (A.35) on the generic connection, Γ_{ABC} (A.30), would eliminate the torsion, and hence reduce Γ_{ABC} to Γ_{ABC}^0 .

The two symmetric properties of the torsionless connection, (A.23) and (A.34), enable us to replace the ordinary derivatives in the definition of the generalized Lie derivative (1.3) by the torsion free semi-covariant derivative, $\nabla_A^0 = \partial_A + \Gamma_A^0$ [23, 24]. In a way, the torsionless connection, Γ_{ABC}^0 (A.31), is the DFT analogy of the Christoffel connection in Riemannian geometry.

It is also worth while to note, upon the section condition (1.2),

$$P_I{}^A \bar{P}_J{}^B \Gamma_{AB}^C \partial_C \simeq 0. \tag{A.36}$$

The usual field strengths of the three connections,

$$\begin{aligned}R_{CDAB} &= \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^E \Gamma_{BED} - \Gamma_{BC}{}^E \Gamma_{AED}, \\ F_{ABpq} &= \partial_A \Phi_{Bpq} - \partial_B \Phi_{Apq} + \Phi_{Apr} \Phi_B{}^r{}_q - \Phi_{Bpr} \Phi_A{}^r{}_q, \\ \bar{F}_{AB\bar{p}\bar{q}} &= \partial_A \bar{\Phi}_{B\bar{p}\bar{q}} - \partial_B \bar{\Phi}_{A\bar{p}\bar{q}} + \bar{\Phi}_{A\bar{p}\bar{r}} \bar{\Phi}_B{}^{\bar{r}}{}_{\bar{q}} - \bar{\Phi}_{B\bar{p}\bar{r}} \bar{\Phi}_A{}^{\bar{r}}{}_{\bar{q}},\end{aligned}\tag{A.37}$$

are, from $[\mathcal{D}_A, \mathcal{D}_B]V_{Cp} = 0$ and $[\mathcal{D}_A, \mathcal{D}_B]\bar{V}_{C\bar{p}} = 0$, related to each other through

$$R_{ABCD} = F_{CDpq}V_A^pV_B^q + \bar{F}_{CD\bar{p}\bar{q}}\bar{V}_A^{\bar{p}}\bar{V}_B^{\bar{q}}. \quad (\text{A.38})$$

It follows then that

$$R_{ABCD} = R_{[AB][CD]}, \quad P_A^I \bar{P}_B^J R_{IJCD} = 0. \quad (\text{A.39})$$

However, none of the field strengths in (A.37) is double-gauge covariant [23].

In order to construct DFT covariant curvature tensors, it is necessary to first define [23],

$$S_{ABCD} := \frac{1}{2} (R_{ABCD} + R_{CDAB} - \Gamma_{AB}^E \Gamma_{ECD}) . \quad (\text{A.40})$$

This rank-four field satisfies, precisely the same symmetric properties as the Riemann curvature,

$$S_{ABCD} = \frac{1}{2} (S_{[AB][CD]} + S_{[CD][AB]}) , \quad (\text{A.41})$$

as well as an additional identity [25],

$$P_I^A \bar{P}_J^B P_K^C \bar{P}_L^D S_{ABCD} \simeq 0. \quad (\text{A.42})$$

The latter holds up to the section condition (1.2) and further implies

$$P^{AC} \bar{P}^{BD} S_{ABCD} = -\frac{1}{2} P^{AC} \bar{P}^{BD} \Gamma_{AB}^E \Gamma_{ECD} \simeq 0. \quad (\text{A.43})$$

Another crucial property of S_{ABCD} is that, under arbitrary variation of the connection, $\delta\Gamma_{ABC}$, it transforms as

$$\delta S_{ABCD} = \mathcal{D}_{[A} \delta\Gamma_{B]CD} + \mathcal{D}_{[C} \delta\Gamma_{D]AB} - \frac{3}{2} \Gamma_{[ABE]} \delta\Gamma_{CD}^E - \frac{3}{2} \Gamma_{[CDE]} \delta\Gamma_{AB}^E. \quad (\text{A.44})$$

Further, from (A.30), if we write

$$\Gamma_{ABC} = \Gamma_{ABC}^0 + \Lambda_{ABC}, \quad \Lambda_{ABC} = \Delta_{Apq} V_B^p V_C^q + \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}_B^{\bar{p}} \bar{V}_C^{\bar{q}}, \quad (\text{A.45})$$

we get

$$S_{ABCD} = S_{ABCD}^0 + \mathcal{D}_{[A}^0 \Lambda_{B]CD} + \mathcal{D}_{[C}^0 \Lambda_{D]AB} + \Lambda_{D[A}^E \Lambda_{|C|B]E} + \Lambda_{B[C}^E \Lambda_{|A|D]E} - \frac{1}{2} \Lambda_{AB}^E \Lambda_{ECD}. \quad (\text{A.46})$$

Consequently, with

$$S_{AB} := S_{ACB}^C, \quad (\text{A.47})$$

we also obtain⁷

$$S_{AB}^0 = S_{AB} + \mathcal{D}_C \Lambda_{(AB)}^C + \frac{1}{2} \Lambda_{CAD} \Lambda_B^C{}^D - \Lambda_{CAD} \Lambda_B^D{}^C. \quad (\text{A.48})$$

⁷Note that, in contrast to (A.46), we have organized the right hand side of the equality in (A.48) as torsionful objects.

Especially for the torsion free case, we have in addition to (A.41) and (A.42) [23]

$$\begin{aligned}
P_I^A P_J^B \bar{P}_K^C \bar{P}_L^D S_{ABCD}^0 &\simeq 0, \\
P_I^A \bar{P}_J^C (P^{BD} - \bar{P}^{BD}) S_{ABCD}^0 &\simeq 0, \\
S^{0A}{}_A &\simeq 0, \\
(P^{AB} P^{CD} + \bar{P}^{AB} \bar{P}^{CD}) S_{ACBD}^0 &\simeq 0, \\
S_{ABCD}^0 + S_{BCAD}^0 + S_{CABD}^0 &= 0 \quad : \quad \text{Bianchi identity},
\end{aligned} \tag{A.49}$$

and the relation (A.44) reduces to

$$\delta S_{ABCD}^0 = \mathcal{D}_{[A}^0 \delta \Gamma_{B]CD}^0 + \mathcal{D}_{[C}^0 \delta \Gamma_{D]AB}^0. \tag{A.50}$$

The variation of the torsionless connection ought to be induced by the (arbitrary) variations of the projections and the DFT-dilaton [23],

$$\begin{aligned}
\delta \Gamma_{CAB}^0 &= 2P_{[A}^D \bar{P}_{B]}^E \nabla_C^0 \delta P_{DE} + 2(\bar{P}_{[A}^D \bar{P}_{B]}^E - P_{[A}^D P_{B]}^E) \nabla_D^0 \delta P_{EC} \\
&\quad - \frac{4}{D-1} (\bar{P}_{C[A} \bar{P}_{B]}^D + P_{C[A} P_{B]}^D) (\partial_D \delta d + P_{E[G} \nabla^{0G} \delta P_{D]}^E) \\
&\quad - \Gamma_{FDE}^0 \delta (\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{FDE},
\end{aligned} \tag{A.51}$$

where the variations of the projections meet [22]

$$\delta P = P \delta P \bar{P} + \bar{P} \delta P P, \quad \delta \bar{P} = P \delta \bar{P} \bar{P} + \bar{P} \delta \bar{P} P, \quad \delta P = -\delta \bar{P}, \tag{A.52}$$

and may be generated by those of the double-vielbein,

$$\delta P_{AB} = V_B^p \delta V_{Ap} + V_A^p \delta V_{Bp}, \quad \delta \bar{P}_{AB} = \bar{V}_B^{\bar{p}} \delta \bar{V}_{A\bar{p}} + \bar{V}_A^{\bar{p}} \delta \bar{V}_{B\bar{p}}. \tag{A.53}$$

Further, the arbitrary variations of the double-vielbein satisfy

$$\delta V_{Ap} = \bar{P}_A^B \delta V_{Bp} + V_A^q \delta V_{B[p} V_{q]}^B, \quad \delta \bar{V}_{A\bar{p}} = P_A^B \delta \bar{V}_{B\bar{p}} + \bar{V}_A^{\bar{q}} \delta \bar{V}_{B[\bar{p}} \bar{V}_{\bar{q}]}^B, \tag{A.54}$$

and the generic torsionful spin connections transform as

$$\begin{aligned}
\delta \Phi_{Apq} &= \mathcal{D}_A (V_p^B \delta V_{Bq}) + V_p^B V_q^C \delta \Gamma_{ABC}, \\
\delta \bar{\Phi}_{A\bar{p}\bar{q}} &= \mathcal{D}_A (\bar{V}_{\bar{p}}^B \delta \bar{V}_{B\bar{q}}) + \bar{V}_{\bar{p}}^B \bar{V}_{\bar{q}}^C \delta \Gamma_{ABC},
\end{aligned} \tag{A.55}$$

and the gravitinos vary as

$$\begin{aligned}\delta\psi_A &= (\delta\psi_{\bar{p}} + \psi_{\bar{q}}\delta\bar{V}_B^{\bar{q}}\bar{V}^B_{\bar{p}})\bar{V}_A^{\bar{p}} - \psi_B\delta V_p^B V_A^p, \\ \delta\psi'_A &= (\delta\psi'_p + \psi'_q\delta V_B^q V^B_p) V_A^p - \psi'_B\delta\bar{V}^B_{\bar{p}}\bar{V}_A^{\bar{p}}.\end{aligned}\tag{A.56}$$

A.3 Projection-aided covariant derivatives and covariant curvatures

Under double-gauge transformations, the connection and the semi-covariant derivative transform as

$$\begin{aligned}(\delta_X - \hat{\mathcal{L}}_X)\Gamma_{CAB} &\simeq 2[(\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{FDE} - \delta_C^F \delta_A^D \delta_B^E] \partial_F \partial_{[D} X_{E]}, \\ (\delta_X - \hat{\mathcal{L}}_X)\nabla_C T_{A_1 \dots A_n} &\simeq \sum_{i=1}^n 2(\mathcal{P} + \bar{\mathcal{P}})_{CA_i}{}^{BFDE} \partial_F \partial_{[D} X_{E]} T_{A_1 \dots A_{i-1} B A_{i+1} \dots A_n}.\end{aligned}\tag{A.57}$$

Hence, the semi-covariant derivative is not generically double-gauge covariant.⁸ We say, a tensor is double-gauge covariant if and only if its double-gauge transformation agrees with the generalized Lie derivative, *i.e.* ‘ $\delta_X \simeq \hat{\mathcal{L}}_X$ ’.

Similarly, while the derivative $D_A = \partial_A + \Phi_A + \bar{\Phi}_A$ (A.19) is $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ local Lorentz covariant, it is not double-gauge covariant, since

$$\begin{aligned}(\delta_X - \hat{\mathcal{L}}_X)\Phi_{Apq} &\simeq 2\mathcal{P}_{ABC}{}^{DEF} \partial_D \partial_{[E} X_{F]} V_p^B V_q^C, \\ (\delta_X - \hat{\mathcal{L}}_X)\bar{\Phi}_{A\bar{p}\bar{q}} &\simeq 2\bar{\mathcal{P}}_{ABC}{}^{DEF} \partial_D \partial_{[E} X_{F]} \bar{V}_{\bar{p}}^B \bar{V}_{\bar{q}}^C.\end{aligned}\tag{A.58}$$

Also from

$$(\delta_X - \hat{\mathcal{L}}_X)S_{ABCD}^0 \simeq 2\mathcal{D}_{[A}^0 ((\mathcal{P} + \bar{\mathcal{P}})_{B][CD]}{}^{EFG} \partial_E \partial_{[F} X_{G]}) + 2\mathcal{D}_{[C}^0 ((\mathcal{P} + \bar{\mathcal{P}})_{D][AB]}{}^{EFG} \partial_E \partial_{[F} X_{G]}) ,\tag{A.59}$$

we see that S_{ABCD}^0 is not double-gauge covariant as well.

However, the characteristic feature of the ‘semi-covariant’ derivative is that —as the name indicates— combined with the projections, it can generate various fully covariant quantities, with respect to double-gauge, $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ double local Lorentz and $\mathbf{O}(D, D)$ symmetries.

⁸Nevertheless exceptionally, Eqs.(A.21, A.22, A.27, A.28, A.34, A.35) are double-gauge covariant. This fact is consistent with the uniqueness of the torsionless connection and the covariant property of the torsion.

We write down (projected) parts of spin connections which are *double-gauge covariant* [23, 24],

$$\bar{P}_A{}^B \Phi_{Bpq}, \quad P_A{}^B \bar{\Phi}_{B\bar{p}\bar{q}}, \quad \Phi_{A[pq} V^A{}_{r]}, \quad \bar{\Phi}_{A[\bar{p}\bar{q}} \bar{V}^A{}_{\bar{r}}], \quad \Phi_{Apq} V^{Ap}, \quad \bar{\Phi}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}}. \quad (\text{A.60})$$

From these, fully covariant quantities follow.

- **Covariant derivatives for $\mathbf{O}(D, D)$ tensors** [23]:

$$\begin{aligned} & P_C{}^D \bar{P}_{A_1}{}^{B_1} \bar{P}_{A_2}{}^{B_2} \dots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \dots B_n}, \\ & \bar{P}_C{}^D P_{A_1}{}^{B_1} P_{A_2}{}^{B_2} \dots P_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \dots B_n}, \\ & P^{AB} \bar{P}_{C_1}{}^{D_1} \bar{P}_{C_2}{}^{D_2} \dots \bar{P}_{C_n}{}^{D_n} \nabla_A T_{B D_1 D_2 \dots D_n}, \\ & \bar{P}^{AB} P_{C_1}{}^{D_1} P_{C_2}{}^{D_2} \dots P_{C_n}{}^{D_n} \nabla_A T_{B D_1 D_2 \dots D_n}, \\ & P^{AB} \bar{P}_{C_1}{}^{D_1} \bar{P}_{C_2}{}^{D_2} \dots \bar{P}_{C_n}{}^{D_n} \nabla_A \nabla_B T_{D_1 D_2 \dots D_n}, \\ & \bar{P}^{AB} P_{C_1}{}^{D_1} P_{C_2}{}^{D_2} \dots P_{C_n}{}^{D_n} \nabla_A \nabla_B T_{D_1 D_2 \dots D_n}. \end{aligned} \quad (\text{A.61})$$

- **Covariant derivatives for $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ tensors**:

$$\begin{aligned} & \mathcal{D}_p T_{\bar{q}_1 \bar{q}_2 \dots \bar{q}_n}, & \mathcal{D}_{\bar{p}} T_{q_1 q_2 \dots q_n}, \\ & \mathcal{D}^p T_{p \bar{q}_1 \bar{q}_2 \dots \bar{q}_n}, & \mathcal{D}^{\bar{p}} T_{\bar{p} q_1 q_2 \dots q_n}, \\ & \mathcal{D}_p \mathcal{D}^p T_{\bar{q}_1 \bar{q}_2 \dots \bar{q}_n}, & \mathcal{D}_{\bar{p}} \mathcal{D}^{\bar{p}} T_{q_1 q_2 \dots q_n}, \end{aligned} \quad (\text{A.62})$$

where we set

$$\mathcal{D}_p := V^A{}_p \mathcal{D}_A, \quad \mathcal{D}_{\bar{p}} := \bar{V}^A{}_{\bar{p}} \mathcal{D}_A. \quad (\text{A.63})$$

These are simply the pull-back of the chiral and anti-chiral $\mathbf{O}(D, D)$ vector indices in (A.61) to the $\mathbf{Spin}(1, D-1)_L$ and $\mathbf{Spin}(D-1, 1)_R$ vector indices using the double-vielbeins.

- **Covariant Dirac operators for fermions, $\rho^\alpha, \psi_{\bar{p}}^\alpha, \rho'^{\bar{\alpha}}, \psi_p'^{\bar{\alpha}}$** [24, 25]:

$$\begin{aligned} & \gamma^p \mathcal{D}_p \rho = \gamma^A \mathcal{D}_A \rho, & \gamma^p \mathcal{D}_p \psi_{\bar{p}} = \gamma^A \mathcal{D}_A \psi_{\bar{p}}, \\ & \mathcal{D}_{\bar{p}} \rho, & \mathcal{D}_{\bar{p}} \psi^{\bar{p}} = \mathcal{D}_A \psi^A, \\ & \bar{\psi}^A \gamma_p (\mathcal{D}_A \psi_{\bar{q}} - \frac{1}{2} \mathcal{D}_{\bar{q}} \psi_A), \end{aligned} \quad (\text{A.64})$$

and⁹

$$\begin{aligned}
\bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}} \rho' &= \bar{\gamma}^A \mathcal{D}_A \rho', & \bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}} \psi'_p &= \bar{\gamma}^A \mathcal{D}_A \psi'_p, \\
\mathcal{D}_p \rho' &, & \mathcal{D}_p \psi'^p &= \mathcal{D}_A \psi'^A, \\
\bar{\psi}'^A \bar{\gamma}_{\bar{p}} (\mathcal{D}_A \psi'_q - \tfrac{1}{2} \mathcal{D}_q \psi'_A) &.
\end{aligned} \tag{A.65}$$

- **Covariant derivatives for $\text{Spin}(1, D-1)_L \times \text{Spin}(D-1, 1)_R$ bi-fundamental spinor, $\mathcal{T}^\alpha_{\bar{\beta}}$:**

$$\gamma^A \mathcal{D}_A \mathcal{T}, \quad \mathcal{D}_A \mathcal{T} \bar{\gamma}^A. \tag{A.66}$$

These are new results we report in this paper. Combining the two, we further define

$$\begin{aligned}
\mathcal{D}_+ \mathcal{T} &:= \gamma^A \mathcal{D}_A \mathcal{T} + \gamma^{(D+1)} \mathcal{D}_A \mathcal{T} \bar{\gamma}^A, \\
\mathcal{D}_- \mathcal{T} &:= \gamma^A \mathcal{D}_A \mathcal{T} - \gamma^{(D+1)} \mathcal{D}_A \mathcal{T} \bar{\gamma}^A.
\end{aligned} \tag{A.67}$$

As shown in section 2.2, for the torsion free case, the corresponding operators are *nilpotent*, up to the section condition (1.2),

$$(\mathcal{D}_+^0)^2 \mathcal{T} \simeq 0, \quad (\mathcal{D}_-^0)^2 \mathcal{T} \simeq 0, \tag{A.68}$$

and hence, they define cohomology.

It is worth while to note

$$\mathcal{D}_\pm (\gamma^{(D+1)} \mathcal{T}) = -\gamma^{(D+1)} \mathcal{D}_\mp \mathcal{T}, \quad \mathcal{D}_\pm (\mathcal{T} \bar{\gamma}^{(D+1)}) = (\mathcal{D}_\mp \mathcal{T}) \bar{\gamma}^{(D+1)}. \tag{A.69}$$

⁹Writing explicitly,

$$\begin{aligned}
\mathcal{D}_A \psi_{\bar{p}} &= (\partial_A + \tfrac{1}{4} \Phi_{Apq} \gamma^{pq}) \psi_{\bar{p}} + \bar{\Phi}_{A\bar{p}}^{\bar{q}} \psi_{\bar{q}}, & \mathcal{D}_A \psi_B &= (\partial_A + \tfrac{1}{4} \Phi_{Apq} \gamma^{pq}) \psi_B + \Gamma_{AB}^C \psi_C, \\
\mathcal{D}_A \psi'_p &= (\partial_A + \tfrac{1}{4} \bar{\Phi}_{A\bar{p}\bar{q}} \bar{\gamma}^{\bar{p}\bar{q}}) \psi'_p + \Phi_{Ap}^q \psi'_q, & \mathcal{D}_A \psi'_B &= (\partial_A + \tfrac{1}{4} \bar{\Phi}_{A\bar{p}\bar{q}} \bar{\gamma}^{\bar{p}\bar{q}}) \psi'_B + \Gamma_{AB}^C \psi'_C.
\end{aligned}$$

- **Covariant curvatures** [23, 25]:

Scalar curvatures,

$$P^{AB}P^{CD}S_{ACBD}, \quad \bar{P}^{AB}\bar{P}^{CD}S_{ACBD}, \quad P^{AB}S_{AB}, \quad \bar{P}^{AB}S_{AB}, \quad (\text{A.70})$$

and a rank-two curvature,

$$S_{p\bar{q}} + \frac{1}{2}\mathcal{D}_{\bar{r}}\bar{\Delta}_{p\bar{q}}^{\bar{r}} + \frac{1}{2}\mathcal{D}_r\Delta_{\bar{q}p}^r, \quad (\text{A.71})$$

where we set

$$S_{p\bar{q}} := V^A{}_p \bar{V}^B{}_{\bar{q}} S_{AB}. \quad (\text{A.72})$$

We emphasize that —while $S_{p\bar{q}}^0$ is covariant for the torsionless connection— if nontrivial torsion is present, $S_{p\bar{q}}$ alone is not covariant: the full expression in Eq.(A.71) is called upon as for a covariant quantity.¹⁰

It is worth while to note, up to the section condition (1.2),

$$\begin{aligned} P^{AB}P^{CD}S_{ACBD} &\simeq P^{AB}S_{AB}, \\ \bar{P}^{AB}\bar{P}^{CD}S_{ACBD} &\simeq \bar{P}^{AB}S_{AB}. \end{aligned} \quad (\text{A.73})$$

Further, in the torsion free case, all the scalar curvatures are equivalent as

$$P^{AB}P^{CD}S_{ACBD}^0 \simeq P^{AB}S_{AB}^0 \simeq -\bar{P}^{AB}\bar{P}^{CD}S_{ACBD}^0 \simeq -\bar{P}^{AB}S_{AB}^0. \quad (\text{A.74})$$

While any of them may constitute the DFT Lagrangian for the NS-NS sector [23, 25], only the following combination allows for the 1.5 formalism to work in supersymmetric double field theory [25],

$$\mathcal{L}_{\text{NSNS}} = \frac{1}{8}e^{-2d}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD}. \quad (\text{A.75})$$

A.4 Reduction to Riemannian geometry in D dimension

Assuming that the upper half blocks are non-degenerate, the double-vielbein satisfying the defining properties (A.7) takes the following most general form [23],

$$V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_p{}^\mu \\ (B+e)_{\nu p} \end{pmatrix}, \quad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{p}}{}^\mu \\ (B+\bar{e})_{\nu\bar{p}} \end{pmatrix}. \quad (\text{A.76})$$

¹⁰For example, see the equations of motion in $\mathcal{N} = 1$ SDFT [25].

Here $e_\mu{}^p$ and $\bar{e}_\nu{}^{\bar{p}}$ are two copies of the D -dimensional vielbein corresponding to the same spacetime metric,

$$e_\mu{}^p e_\nu{}^q \eta_{pq} = -\bar{e}_\mu{}^{\bar{p}} \bar{e}_\nu{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}} = g_{\mu\nu}, \quad (\text{A.77})$$

and $B_{\mu\nu}$ corresponds to the Kalb-Ramond two-form gauge field. We also set in (A.76),

$$B_{\mu p} = B_{\mu\nu} (e^{-1})_p{}^\nu, \quad B_{\mu \bar{p}} = B_{\mu\nu} (\bar{e}^{-1})_{\bar{p}}{}^\nu. \quad (\text{A.78})$$

It is worth while to note that, $(\bar{e}^{-1}e)_{\bar{p}}{}^p$ and $(e^{-1}\bar{e})_p{}^{\bar{p}}$ are local Lorentz transformations, such that

$$(\bar{e}^{-1}e)_{\bar{p}}{}^p (\bar{e}^{-1}e)_{\bar{q}}{}^q \eta_{pq} = -\bar{\eta}_{\bar{p}\bar{q}}, \quad (e^{-1}\bar{e})_p{}^{\bar{p}} (e^{-1}\bar{e})_q{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}} = -\eta_{pq}. \quad (\text{A.79})$$

Instead of (A.76), we may choose the following alternative parametrization,

$$V_A{}^p = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \tilde{e})^{\mu p} \\ (\tilde{e}^{-1})^p{}_\nu \end{pmatrix}, \quad \bar{V}_A{}^{\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\beta + \tilde{\bar{e}})^{\mu \bar{p}} \\ (\tilde{\bar{e}}^{-1})^{\bar{p}}{}_\nu \end{pmatrix}, \quad (\text{A.80})$$

with

$$\beta^{\mu p} = \beta^{\mu\nu} (\tilde{e}^{-1})^p{}_\nu, \quad \beta^{\mu \bar{p}} = \beta^{\mu\nu} (\tilde{\bar{e}}^{-1})^{\bar{p}}{}_\nu. \quad (\text{A.81})$$

Physically, $\tilde{e}^\mu{}_p$ and $\tilde{\bar{e}}^\mu{}_{\bar{p}}$ correspond to a pair of T-dual vielbeins, such that both give rise to the winding mode spacetime metric,

$$\tilde{e}^\mu{}_p \tilde{e}^\nu{}_q \eta^{pq} = -\tilde{\bar{e}}^\mu{}_{\bar{p}} \tilde{\bar{e}}^\nu{}_{\bar{q}} \bar{\eta}^{\bar{p}\bar{q}} = (g - B g^{-1} B)^{-1 \mu\nu}. \quad (\text{A.82})$$

Note that, in the winding mode sector, the D -dimensional curved spacetime indices are all upside-down, such as \tilde{x}_μ , $\tilde{e}^\mu{}_p$, $\tilde{\bar{e}}^\mu{}_{\bar{p}}$, $\beta^{\mu\nu}$ (cf. x^μ , $e_\mu{}^p$, $\bar{e}_\mu{}^{\bar{p}}$, $B_{\mu\nu}$).

In connection to the section condition, $\partial^A \partial_A \simeq 0$ (1.2), the former parametrization (A.76) matches well with the choice, $\frac{\partial}{\partial \tilde{x}_\mu} \simeq 0$, while the latter is natural when $\frac{\partial}{\partial x^\mu} \simeq 0$. Yet if we consider dimensional reductions from D to lower dimensions, there is no longer preferred parametrization. For related discussions we refer to [40–42, 53–56].

Henceforth we restrict ourselves to the parametrization (A.76) and to the section choice,¹¹

$$\frac{\partial}{\partial \tilde{x}_\mu} \simeq 0. \quad (\text{A.83})$$

¹¹This restriction reduces our $\mathbf{O}(D, D)$ covariant stringy differential geometry to the generalized geometry.

In analogy to the DFT master semi-covariant derivative, \mathcal{D}_A (A.18), we consider a genuinely D -dimensional master derivative [23],

$$D_\mu = \nabla_\mu + \omega_\mu + \bar{\omega}_\mu, \quad (\text{A.84})$$

which is covariant with respect to the D -dimensional diffeomorphism and the pair of local Lorentz groups, $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$, as ∇_μ denotes the standard D -dimensional diffeomorphism covariant derivative, while ω_μ and $\bar{\omega}_\mu$ correspond to the spin connections of the local Lorentz groups, $\mathbf{Spin}(1, D-1)_L$ and $\mathbf{Spin}(D-1, 1)_R$ respectively. Yet, it is not $\mathbf{O}(D, D)$ covariant.

It satisfies

$$D_\lambda g_{\mu\nu} = \nabla_\lambda g_{\mu\nu} = 0, \quad D_\mu e_{\nu m} = 0, \quad D_\mu \bar{e}_{\nu \bar{n}} = 0, \quad (\text{A.85})$$

and hence, as in (A.29), the D -dimensional spin connections are determined by

$$\omega_{\mu mn} = (e^{-1})_m{}^\nu \nabla_\mu e_{\nu n}, \quad \bar{\omega}_{\mu \bar{m} \bar{n}} = (\bar{e}^{-1})_{\bar{m}}{}^\nu \nabla_\mu \bar{e}_{\nu \bar{n}}. \quad (\text{A.86})$$

For the diffeomorphism covariant derivative, ∇_μ , we assume the torsionless Christoffel connection,

$$\left\{ \begin{smallmatrix} \lambda \\ \mu \quad \nu \end{smallmatrix} \right\} = \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}). \quad (\text{A.87})$$

In terms of the parametrization of the double-vielbein (A.76), the projection-aided covariant spin connections in Eq.(A.60) read explicitly for the torsionless connection,

$$\begin{aligned} \sqrt{2} \bar{V}^A{}_{\bar{p}} \Phi_{Aqr}^0 &\simeq \bar{e}^\mu{}_{\bar{p}} \omega_{\mu qr} + \frac{1}{2} H_{\bar{p}qr}, & \sqrt{2} V^A{}_p \bar{\Phi}_{A\bar{q}\bar{r}}^0 &\simeq e^\mu{}_p \bar{\omega}_{\mu \bar{q}\bar{r}} + \frac{1}{2} H_{p\bar{q}\bar{r}}, \\ \sqrt{2} \Phi_{A[pq}^0 V^A{}_{r]} &\simeq \omega_{\mu[pq} e^\mu{}_{r]} + \frac{1}{6} H_{pqr}, & \sqrt{2} \bar{\Phi}_{A[\bar{p}\bar{q}}^0 \bar{V}^A{}_{\bar{r}]} &\simeq \bar{\omega}_{\mu[\bar{p}\bar{q}} \bar{e}^\mu{}_{\bar{r}]} + \frac{1}{6} H_{\bar{p}\bar{q}\bar{r}}, \\ \sqrt{2} \bar{V}^{Ap} \Phi_{Apq}^0 &\simeq e^{\mu p} \omega_{\mu pq} - 2\partial_q \phi, & \sqrt{2} \bar{V}^{A\bar{p}} \bar{\Phi}_{A\bar{p}\bar{q}}^0 &\simeq \bar{e}^{\mu \bar{p}} \bar{\omega}_{\mu \bar{p}\bar{q}} - 2\partial_{\bar{q}} \phi. \end{aligned} \quad (\text{A.88})$$

Clearly, these are diffeomorphism and B -field gauge symmetry covariant, and hence, as asserted, double-gauge covariant. Using the results, we may express all the fully covariant derivatives in section A.3 in terms of the usual D -dimensional Riemannian terminology [23, 24]. For example, for the fermions, ρ^α , $\psi_{\bar{p}}^\alpha$, we get

$$\begin{aligned} \sqrt{2} \gamma^A \mathcal{D}_A \rho &\simeq \gamma^m (\partial_m \rho + \frac{1}{4} \omega_{mnp} \gamma^{np} \rho + \frac{1}{24} H_{mnp} \gamma^{np} \rho - \partial_m \phi \rho), \\ \sqrt{2} \gamma^A \mathcal{D}_A \psi_{\bar{p}} &\simeq \gamma^m (\partial_m \psi_{\bar{p}} + \frac{1}{4} \omega_{mnp} \gamma^{np} \psi_{\bar{p}} + \bar{\omega}_{m\bar{p}\bar{q}} \psi^{\bar{q}} + \frac{1}{24} H_{mnp} \gamma^{np} \psi_{\bar{p}} + \frac{1}{2} H_{m\bar{p}\bar{q}} \psi^{\bar{q}} - \partial_m \phi \psi_{\bar{p}}), \\ \sqrt{2} \bar{V}^A{}_{\bar{p}} \mathcal{D}_A \rho &\simeq \partial_{\bar{p}} \rho + \frac{1}{4} \omega_{\bar{p}qr} \gamma^{qr} \rho + \frac{1}{8} H_{\bar{p}qr} \gamma^{qr} \rho, \\ \sqrt{2} \mathcal{D}_A \psi^A &\simeq \partial^{\bar{p}} \psi_{\bar{p}} + \frac{1}{4} \omega_{\bar{p}qr} \gamma^{qr} \psi^{\bar{p}} + \bar{\omega}_{\bar{p}\bar{q}} \psi^{\bar{q}} + \frac{1}{8} H_{\bar{p}qr} \gamma^{qr} \psi^{\bar{p}} - 2\partial_{\bar{p}} \phi \psi^{\bar{p}}, \end{aligned} \quad (\text{A.89})$$

and for the other fermions, $\rho'^{\bar{\alpha}}, \psi'_p{}^{\bar{\alpha}}$, which are in the opposite $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ representations, we have

$$\begin{aligned}
\sqrt{2}\bar{\gamma}^A \mathcal{D}_A^0 \rho' &\simeq \bar{\gamma}^{\bar{m}} \left(\partial_{\bar{m}} \rho' + \frac{1}{4} \bar{\omega}_{\bar{m}\bar{n}\bar{p}} \bar{\gamma}^{\bar{n}\bar{p}} \rho' + \frac{1}{24} H_{\bar{m}\bar{n}\bar{p}} \bar{\gamma}^{\bar{n}\bar{p}} \rho' - \partial_{\bar{m}} \phi \rho' \right), \\
\sqrt{2}\bar{\gamma}^A \mathcal{D}_A^0 \psi'_p &\simeq \bar{\gamma}^{\bar{m}} \left(\partial_{\bar{m}} \psi'_p + \frac{1}{4} \bar{\omega}_{\bar{m}\bar{n}\bar{p}} \bar{\gamma}^{\bar{n}\bar{p}} \psi'_p + \omega_{\bar{m}pq} \psi'^q + \frac{1}{24} H_{\bar{m}\bar{n}\bar{p}} \bar{\gamma}^{\bar{n}\bar{p}} \psi'_p + \frac{1}{2} H_{\bar{m}pq} \psi'^q - \partial_{\bar{m}} \phi \psi'_p \right), \\
\sqrt{2}V^A{}_p \mathcal{D}_A^0 \rho' &\simeq \partial_p \rho' + \frac{1}{4} \bar{\omega}_{p\bar{q}\bar{r}} \bar{\gamma}^{\bar{q}\bar{r}} \rho' + \frac{1}{8} H_{p\bar{q}\bar{r}} \bar{\gamma}^{\bar{q}\bar{r}} \rho', \\
\sqrt{2}\mathcal{D}_A^0 \psi'^A &\simeq \partial^p \psi'_p + \frac{1}{4} \bar{\omega}_{p\bar{q}\bar{r}} \bar{\gamma}^{\bar{q}\bar{r}} \psi'^p + \omega^p{}_{pq} \psi'^q + \frac{1}{8} H_{p\bar{q}\bar{r}} \bar{\gamma}^{\bar{q}\bar{r}} \psi'^p - 2\partial_p \phi \psi'^p.
\end{aligned} \tag{A.90}$$

Here, for simplicity, we set $\partial_p = (e^{-1})_p{}^\mu \partial_\mu$, $\partial_{\bar{p}} = (\bar{e}^{-1})_{\bar{p}}{}^\mu \partial_\mu$, *etc.*

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